

**Parameter-Dependent Lyapunov Functions and  
Stability Analysis of Linear Parameter-Dependent  
Dynamical Systems**

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**Parameter-Dependent Lyapunov Functions and  
Stability Analysis of Linear Parameter-Dependent  
Dynamical Systems**

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*To my family for their love and understanding*

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## Notation and Acronyms

$\mathbb{R}$	Real numbers
$\mathbb{R}^n$	Real column vectors with $n$ entries
$\mathbb{R}^{m \times n}$	Real matrices of dimension $m \times n$
$\mathbb{Z}_+^0$	Nonegative integer numbers $\{0, 1, 2, \dots\}$
$j$	The imaginary unit, i.e. $j = \sqrt{-1}$
$\otimes \oplus$	Kronecker product and sum
$\star$	Bialternate product
$\lambda_i(A)$	$i$ th eigenvalue of the matrix $A$
$I_n$	Identity matrix of dimension $n \times n$ (also denoted $I$ when the dimension is clear from the context)
$\text{int}(\mathcal{D})$	Interior of the set $\mathcal{D}$
$\partial\mathcal{D}$	Boundary of the set $\mathcal{D}$
$\mathcal{A}$	Set of Hurwitz matrices
$\bar{A}$	$A \oplus A$ , $A \in \mathbb{R}^{n \times n}$
$\tilde{A}$	$A \star I_n + I_n \star A = 2A \star I_n$ , $A \in \mathbb{R}^{n \times n}$
$\text{mspec}(A)$	Multispectrum of matrix $A$ , i.e. the set consisting of all the eigenvalues of $A$ , including repeated eigenvalues.
$\mathcal{I}_n$	Index set $\{1, 2, \dots, n\}$
$\mathcal{I}_n^0$	Index set $\{0, 1, 2, \dots, n\}$
$\bar{\cup}$	Ordered union of two sets
$\mathcal{D}^\#$	Cardinality of the set $\mathcal{D}$
$\det(A)$	Determinant of the matrix $A$
LTI	Linear time invariant
LPV	Linear parameter varying
LTIPD	Linear time invariant parameter dependent
LMI	Linear matrix inequality
PDLM	Parameter dependent Lyapunov matrix

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## SUMMARY

The purpose of this thesis is to develop new stability conditions for several linear dynamic systems, including linear parameter-varying (LPV), time-delay systems (LPVTD), slow LPV systems, and parameter-dependent linear time invariant (LTI) systems. These stability conditions are less conservative and/or computationally easier to apply than existing ones.

This dissertation is composed of four parts. In the first part, the complete stability domain for LTI parameter-dependent (LTIPD) systems is synthesized by extending existing results in the literature. This domain is calculated through a guardian map which involves the determinant of the Kronecker sum of a matrix with itself. The stability domain is synthesized for both single- and multi-parameter dependent LTI systems. The single-parameter case is easily computable, whereas the multi-parameter case is more involved. The determinant of the bialternate sum of a matrix with itself is also exploited to reduce the computational complexity.

In the second part of the thesis, a class of parameter-dependent Lyapunov functions is proposed, which can be used to assess the stability properties of single-parameter LTIPD systems in a non-conservative manner. It is shown that stability of LTIPD systems is equivalent to the existence of a Lyapunov function of a polynomial type (in terms of the parameter) of known, bounded degree satisfying two matrix inequalities. The bound of polynomial degree of the Lyapunov functions is then reduced by taking advantage of the fact that the Lyapunov matrices are symmetric. If the matrix multiplying the parameter is not full rank, the polynomial order can be reduced even further. It is also shown that checking the feasibility of these matrix inequalities over a compact set can be cast as a convex optimization problem. Such Lyapunov functions and stability conditions for affine

single-parameter LTIPD systems are then generalized to single-parameter polynomially-dependent LTIPD systems and affine multi-parameter LTIPD systems.

The third part of the thesis provides one of the first attempts to derive computationally tractable criteria for analyzing the stability of LPV *time-delayed* systems. It presents both delay-independent and delay-dependent stability conditions, which are derived using appropriately selected Lyapunov-Krasovskii functionals. According to the system parameter dependence, these functionals can be selected to obtain increasingly non-conservative results. Gridding techniques may be used to cast these tests as Linear Matrix Inequalities (LMI's). In cases when the system matrices depend affinely or quadratically on the parameter, gridding may be avoided. These LMI's can be solved efficiently using available software. A numerical example of a time-delayed system motivated by a metal removal process is used to demonstrate the theoretical results.

In the last part of the thesis, topics for future investigation are proposed. Among the most interesting avenues for research in this context, it is proposed to extend the existing stability analysis results to controller synthesis, which will be based on the same Lyapunov functions used to derive the nonconservative stability conditions. While designing the dynamic controller for linear and parameter-dependent systems, it is desired to take the advantage of the rank deficiency of the system matrix multiplying the parameter such that the controller is of lower dimension, or rank deficient without sacrificing the performance of closed-loop systems.

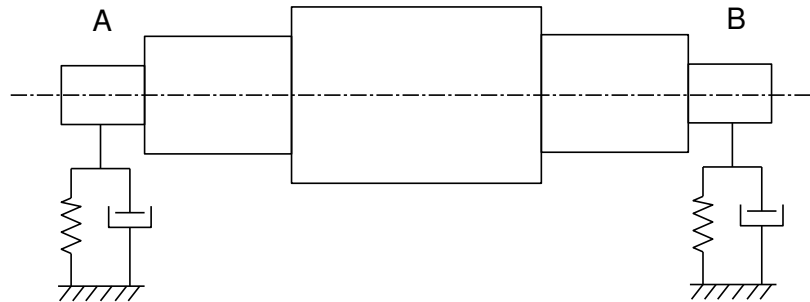
# CHAPTER I

## BACKGROUND AND MOTIVATION

This Thesis has three major contributions and each of them has its own engineering background and theoretical motivation. Below we provide the motivation for investigating the series of research problems addressed here.

### *1.1 Robust Stability of LTI Parameter-Dependent Systems*

This research work is motivated by the need to control active magnetic bearings (AMB) for gyroscopic mechanical systems. The example of a rotating shaft supported on an AMB is one example of an LTIPD system arising from a real-world application. The advantages of AMB's, such as contactless and frictionless operation in normal operation, without lubrication, make them virtually maintenance-free and attractive for various applications [66]. Due to their complex structure, precise design requirements, and increasing application in industry, controller design for AMB's has attracted much attention recently [73, 81, 50, 67, 29]. However, the existing research results have not taken advantage of the special structure of the controlled system, such as for the case of gyroscopic mechanical systems.



**Figure 1:** Supported Flexible Rotor Beam on AMB

The linear model of a rotor shaft supported by AMBs (as in Fig. 1) can be obtained using

a Finite Element Model (FEM) program. Assuming an isotropic rotor, such a model is traditionally described by the equations

$$M\ddot{x} + D\dot{x} + \omega G\dot{y} + Kx = B_1 u_1 \quad (1a)$$

$$M\ddot{y} + D\dot{y} - \omega G\dot{x} + Ky = B_2 u_2 \quad (1b)$$

where  $M$ ,  $K$  and  $D$  are the mass, stiffness, and damping matrices, respectively. The matrix  $G$  is the gyroscopic matrix, which is responsible for the cross-coupling between the  $x - z$  and  $y - z$  planes. The matrix  $M$  is always symmetric positive definite, while the matrices  $K$  and  $D$  are assumed to be symmetric positive-definite and semi-definite respectively. The matrix  $G$  is symmetric and  $\omega$  is the angular velocity of the rotor about its symmetry axis. The matrices in Eqs. (1) are assumed constant for the time being, since we only deal with the linear case.

Let  $m_x = [x \ \dot{x}]^T$  and  $m_y = [y \ \dot{y}]^T$ . The model in Eqs. (1) can be arranged as

$$\frac{d}{dt} \begin{bmatrix} m_x \\ m_y \end{bmatrix} = \begin{bmatrix} A_{m0} & 0 \\ 0 & A_{m0} \end{bmatrix} \begin{bmatrix} m_x \\ m_y \end{bmatrix} + \omega \begin{bmatrix} 0 & -A_g \\ A_g & 0 \end{bmatrix} \begin{bmatrix} m_x \\ m_y \end{bmatrix} + \begin{bmatrix} B_{m1} & 0 \\ 0 & B_{m2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (2)$$

where

$$A_{m0} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad A_g = \begin{bmatrix} 0 & 0 \\ 0 & -M^{-1}G \end{bmatrix},$$

$$B_{m1} = \begin{bmatrix} 0 \\ M^{-1}B_1 \end{bmatrix}, \quad B_{m2} = \begin{bmatrix} 0 \\ M^{-1}B_2 \end{bmatrix}$$

Since the matrix  $M$  is positive definite, its Cholesky square root  $L$  exists, such that  $M = L^T L$ . Defining a new coordinate transformation, we can investigate more closely the structure of the model in Eqs. (1) or (2) and get a better insight. Using the state transformation

$$\begin{bmatrix} \tilde{m}_x \\ \tilde{m}_y \end{bmatrix} = \begin{bmatrix} (L^{-1}KL^{-T})^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & (L^{-1}KL^{-T})^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} L^T & 0 & 0 & 0 \\ 0 & L^T & 0 & 0 \\ 0 & 0 & L^T & 0 \\ 0 & 0 & 0 & L^T \end{bmatrix} \begin{bmatrix} m_x \\ m_y \end{bmatrix} \quad (3)$$

Eq. (2) can be written as

$$\frac{d}{dt} \begin{bmatrix} \tilde{m}_x \\ \tilde{m}_y \end{bmatrix} = \begin{bmatrix} \bar{A}_{m0} & 0 \\ 0 & \bar{A}_{m0} \end{bmatrix} \begin{bmatrix} \tilde{m}_x \\ \tilde{m}_y \end{bmatrix} + \omega \begin{bmatrix} 0 & -\bar{A}_g \\ \bar{A}_g & 0 \end{bmatrix} \begin{bmatrix} \tilde{m}_x \\ \tilde{m}_y \end{bmatrix} + \begin{bmatrix} \bar{B}_{m1} & 0 \\ 0 & \bar{B}_{m2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (4)$$

where

$$\bar{A}_{m0} = \begin{bmatrix} 0 & \bar{K}^{\frac{1}{2}} \\ -\bar{K}^{\frac{1}{2}} & -\bar{D} \end{bmatrix}, \quad \bar{A}_g = \begin{bmatrix} 0 & 0 \\ 0 & -\bar{G} \end{bmatrix} = \bar{A}_g^T \quad (5)$$

and

$$\bar{B}_{m1} = \begin{bmatrix} 0 \\ L^{-1}B_1 \end{bmatrix}, \quad \bar{B}_{m2} = \begin{bmatrix} 0 \\ L^{-1}B_2 \end{bmatrix} \quad (6)$$

with  $\bar{K} = L^{-1}KL^{-T}$ ,  $\bar{D} = L^{-1}DL^{-T}$ ,  $\bar{G} = L^{-1}GL^{-T}$ . The model in Eq. (4) has kept the original properties. That is,  $\bar{K} = \bar{K}^T > 0$ ,  $\bar{D} = \bar{D}^T \geq 0$ ,  $\bar{G} = \bar{G}^T$ . In particular, note that the cross-coupling state-matrix in Eq. (4) is skew-symmetric. In the case  $D$  is small (but non-zero), then  $\bar{A}_{m0}$  is *almost* skew-symmetric.

According to the previous developments [73], the model of an AMB gyroscopic mechanical system can then be summarized as

$$\begin{aligned} \dot{X} &= A_0X + \omega A_gX + BU \\ &= (A_0 + \omega A_g)X + BU \end{aligned} \quad (7)$$

where  $A_0, A_g \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  are constant matrices of proper dimensions as in (4), and  $\omega$  is a constant or sufficiently slowly time-varying scalar parameter. This model has the following two special properties:

- (i) It is a slow LPV system. This is because in most industrial applications, the rotor speed  $\omega$  changes much slower than the beam dynamics.
- (ii)  $A_g$  is typically rank-deficient, i.e.  $\text{rank}(A_g) = r < n$ .

We wish to take advantage of the special structure implied by (i) – (iv) in order to design control laws that are non-conservative in terms of stability robustness and performance.



Motivated by the model in Eq.(7), it is desired first to obtain necessary and sufficient stability conditions for the following slow LPV dynamic model without control input

$$\dot{X} = (A_0 + \rho A_g)X \quad (8)$$

where  $\rho$  is a slowly time-varying or constant parameter, and the matrix  $A_g$  satisfies the property (ii) above.

The stability of systems of the form (8) has been investigated within the framework of linear, time invariant uncertain systems. The need to determine the bounds on system uncertainty that guarantee stability for the perturbed system has been the subject of intensive research in the past several years. Several parameter-dependent Lyapunov functions have been suggested in the literature to find such bounds [37, 12, 35, 40, 44, 56]. However, the use of Lyapunov function methods may give rise to stability conditions that are sufficient but not necessary. References [20] and [21] studied quadratic  $\delta$ -Hurwitz and  $\mathcal{D}$ -stability and gave robust stability conditions for the case of parametric uncertainty. For quadratic stability, Refs. [1] and [46] gave necessary and sufficient conditions which are valid even for time-varying linear systems. However, quadratic stability is, in general, more conservative than robust stability [63, 21]. Saydy *et al.* [64, 65] defined a particular guardian map and used it to study the stability of LTIPD systems of the form

$$\dot{x} = A(\rho)x, \quad A(\rho) = A_0 + \rho A_1 + \rho^2 A_2 + \dots + \rho^m A_m \quad (9)$$

and

$$\dot{x} = A(\rho_1, \rho_2)x, \quad A(\rho_1, \rho_2) = \sum_{i_1, i_2=0}^{i_1+i_2=m} \rho_1^{i_1} \rho_2^{i_2} A_{i_1, i_2}. \quad (10)$$

The guardian map in [64] is the determinant of the Kronecker sum of a matrix with itself. Using this guardian map, Saydy *et al.* gave necessary and sufficient stability conditions with respect to a given parameter domain, for the particular LTIPD systems in (9) and (10). This method was later extended in [7] and [63] to LTI systems with many parameters in the form

$$\dot{x} = A(\rho_1, \rho_2, \dots, \rho_m)x, \quad A(\rho_1, \rho_2, \dots, \rho_m) = A_0 + \sum_{i=1}^m \rho_i A_i. \quad (11)$$

However, the stability conditions in [7] and [63] are only sufficient. Fu and Barmish [24] gave the maximal stability interval around the origin for LTIPD systems of the form (11) with  $m = 1$  and  $A_0$  Hurwitz. Mustafa [54] studied the robust stability problem of LTIPD systems using the bialternate sum of matrices. The determinant of the bialternate sum of a matrix  $A \in \mathbb{R}^{n \times n}$  with itself is not a guardian map. Nonetheless, it can be used in a similar way as the Kronecker sum to guard Hurwitz matrices with minor changes. The advantage of the bialternate sum used by Mustafa is that it involves fewer calculations than the Kronecker sum. This property is also explored in this work to reduce the computations required for the derived stability tests.

The existing results in [64, 65, 7] give necessary and sufficient stability conditions for an *a priori* given single- or multi-parameter interval set. Furthermore, [63] provides a bounded interval set which is only sufficient in guaranteeing the stability of LTIPD systems. A question which arises naturally is how to find the exact stability domain for single- or multi-parameter dependent systems. In fact, the complete stability domain may be composed of one or several pieces of connected sets.

It should be pointed out that the derived conditions in this work can also be used to determine the stability of “slow” LPV systems. As shown in [34], given the system

$$\dot{x} = (A_0 + \rho(t)A_g)x \quad (12)$$

where  $A_0, A_g \in \mathbb{R}^{n \times n}$ ,  $\rho(t) \in [\underline{\rho}, \bar{\rho}]$ ,  $\dot{\rho}(t) \in [\underline{\dot{\rho}}, \bar{\dot{\rho}}]$ ,  $\underline{\dot{\rho}} \leq 0 \leq \bar{\dot{\rho}}$  and there exists  $\epsilon > 0$  such that  $\sup \min\{\|\dot{\rho}(t)\|, \|\ddot{\rho}(t)\|\} < \epsilon$  for all  $t \geq 0$ . If  $\rho(t)$  changes sufficiently slowly and with  $\underline{\dot{\rho}}, \bar{\dot{\rho}}$  being sufficient small, the following conditions are equivalent:

- (i) The system (12) is asymptotically stable.
- (ii)  $\text{Re}[\lambda_i(A_0 + \rho A_g)] < 0$ ,  $\rho \in [\underline{\rho}, \bar{\rho}]$ ,  $i = 1, 2, \dots, n$ .

This implies that the stability of the “slowly-varying” LPV system in (12) can be inferred from the stability of the LTIPD system  $\dot{x} = (A_0 + \rho A_g)x$  where  $\rho$  is unknown but *constant* in the interval  $[\underline{\rho}, \bar{\rho}]$ .

## ***1.2 Parameter-Dependent Lyapunov Functions for LTI Systems***

When a controller is synthesized for an LPV or LTIPD system, high performance is desired under the basic requirement of stability for the closed loop system. Sufficient stability conditions provide only a subset of the whole stability domain, which is characterized by a necessary and sufficient stability condition. A controller synthesized within the whole stability domain or under a necessary and sufficient stability condition will be able to achieve better performance, compared to a controller optimized within a subset of the stability domain or using a sufficient stability condition only. Several types of parameter dependent Lyapunov matrices (PDLM) or functions have been suggested in the literature to perform stability analysis or controller synthesis for LPV systems or LTIPD systems [37, 12, 35, 40, 44, 56, 2, 73]. However, the use of these existing Lyapunov function methods gives rise to stability conditions that are sufficient but not necessary. Therefore such stability conditions may be overly conservative. It is desirable to find a necessary and sufficient stability condition to reduce conservatism.

Lyapunov stability can be applied to both LPV and LTIPD systems and the induced stability conditions depend on the particular Lyapunov function chosen. If the Lyapunov function is fixed, i.e., not parameter-dependent (for the case of LPV or LTIPD systems), the corresponding stability is the so-called quadratic stability. This is a more conservative notion of stability compared to the stability induced by a properly chosen parameter-dependent Lyapunov function. It is expected that parameter-dependent Lyapunov functions can be used to provide necessary and sufficient stability conditions for LPV and LTIPD systems. It is reminded that for the case of stable LTI systems, a Lyapunov function always exists. Moreover, it can be explicitly expressed as an integral of the exponential function of the system matrix [98]. However, this kind of Lyapunov function cannot be used to check the stability of an LTIPD system since it does not exist if the LTIPD system is unstable. Even if the LTIPD system is stable, the Lyapunov inequalities corresponding to this kind of Lyapunov function cannot be expressed in terms of LMIs. In order to check the stability of an

LTIPD system, a proper Lyapunov function is desired, which has a simple form, such that both its positive-definiteness and the corresponding Lyapunov inequalities can be expressed in terms of LMIs of low dimension so that they can be easily checked with current LMI techniques [28, 17].

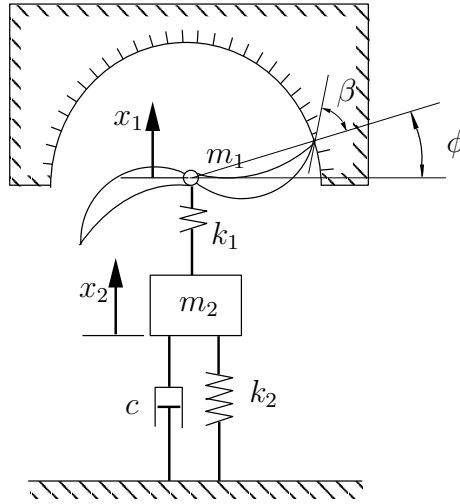
While finding such a Lyapunov function which is both necessary and sufficient for stability, an additional task will be to simplify the Lyapunov function and thus, simplify the stability conditions by taking the advantage of the special property of our LPV or LTIPD system. Especially, for our case the model of an AMB-supported flexible rotor is a slow LPV system in which the system matrix  $A(\rho)$  depends linearly on the rotor speed  $\rho$  by  $A(\rho) = A_0 + \rho A_g$  and the matrix  $A_g$  is rank deficient, i.e,  $\text{rank}(A_g) < n$ , where  $n$  is the dimension of the state.

### ***1.3 Stability Analysis of LPV Time-Delayed Systems***

Time delays are ubiquitous in control systems. Their presence can have a deleterious effect on the system performance and can even lead to instability. The effect of time-delays on LPV systems has not been addressed in great depth in the literature. Apart from its theoretical interest, this work was motivated by the industrial problem of turning metal cutting. Machining of materials is often accompanied by a violent vibration between work-piece and the cutting tool. This damaging vibration is called chatter, which has adverse effects on surface finish, machining accuracy, tool life and machine life [16]. It is of great interest to be able to remove metal during the process as fast as possible, thus increasing the throughput in the production line. Studies have shown, however, that increased cutting speeds have as a result the onset of chatter [43, 16]. To avoid this chatter, in general the machine tool should be operated at low material removal rates.

It is desired to design control laws to reduce or eliminate chatter so that higher metal removing speeds can be achieved [59]. Therefore, there is a need to investigate the dynamics of machining of materials and to analyze their stability. In a milling process, the

work-piece is clamped and fed to a rotating multi-tooth cutter. The teeth of the cutter periodically enter and exit the work-piece, as shown in Fig. 2. The dynamics of this milling process is characterized by a linear parameter varying (LPV) time-delay system as shown in Fig. 12(b) and Eq. (287), where the time delay  $\tau$  is the time interval between successive cuts. The adverse effect of time delay, unavoidably resulting from the regenerative effect of turning metal cutting, can dramatically limit the performance and even destabilize the closed loop system [52]. To achieve higher performance, the controller must be optimized under a nonconservative stability condition, providing more freedom for controller synthesis compared to a conservative condition. Thus, it is important to find a computationally tractable, nonconservative stability conditions for such LPV time-delay systems.



**Figure 2:** A Simple Schematic of a Milling Process

The theory of LPV (non-delayed) systems has witnessed an explosion over the recent years. Stability analysis and synthesis results have been reported, for example, in [11, 4, 83, 9, 30, 49, 31, 79]. The theory of LPV, *time-delayed* systems is less developed, however. Some initial results have been reported in [82]. In that reference, the authors analyzed a time-delayed LPV system where the state matrices and the time delay are functions of time-varying parameters that can be measured in real-time. Their analysis uses a Lyapunov-Krasovskii functional in which the kernel of the integral term is parameter independent. Reference [82] also presents state-feedback controllers for time-delayed LPV systems that guarantee

desired  $L_2$ -gain performance. Despite the limited existing results for the analysis and control synthesis for LPV time-delayed systems, there are several cases where time-delayed systems which depend on parameters arise naturally in applications. In milling, for example, the dynamics of the cutting process involve delayed states as well as time-varying parameters [71]. As previously outlined, this particular application provided the motivation of this work.

For LPV time-delayed systems, such as those characterizing a milling process, it is desirable to find new and less conservative delay-independent and delay-dependent stability criteria. In both cases, these criteria will be obtained by application of well-known Lyapunov-Krasovskii stability results [36]. To reduce conservatism for the delay-independent stability case, in this work we introduce several appropriately constructed Lyapunov-Krasovskii functionals. All the stability tests are given in terms of Linear Matrix Inequalities (LMI's). Typically, the resulting LMI's are infinite-dimensional. Thus, gridding and/or relaxation techniques are used to project these LMI's to finite dimensions [74, 82, 30, 31, 32]. Efficient algorithms can then be used to solve these LMI's.

## ***1.4 Organization of the Thesis***

This thesis includes eight chapters. The first chapter presents the background and motivation of the research. The second chapter is a self-contained chapter and it introduces some basic mathematical concepts used in the subsequent chapters. Three major contributions are achieved and presented from Chapter 3 to Chapter 7. Future work will be discussed in Chapter 8.

The first contribution is introduced in Chapter 3. The complete, exact stability domain for both single- and multi-parameter dependent LTI systems is synthesized by extending existing results, which can only give one stability interval over  $\mathbb{R}$  even though the whole stability domain could be a single interval or a union of several disjointed interval over  $\mathbb{R}$ . The possibility of reducing the complexity of the calculation is also investigated.

The second contribution is developed in Chapters 4, 5 and 6, where a class of parameter-dependent Lyapunov functions is proposed, which can be used to assess the stability properties of parameter-dependent LTI systems in a non-conservative manner. In Chapter 4, it is shown that stability of single-parameter LTIPD systems is equivalent to the existence of a Lyapunov function of a polynomial type (in terms of the parameter) of known, bounded degree. It is also shown that checking the feasibility of the two Lyapunov matrix inequalities over a compact set can be cast as a convex optimization problem. Therefore the nonconservative stability conditions can be cast as two LMIs. Nonconservative stability conditions for affine single-parameter LTIPD systems are then generalized to single-parameter polynomially-dependent LTIPD systems in Chapter 5, and affine multi-parameter LTIPD systems in Chapter 6.

The third contribution of this thesis is on the stability analysis of LPV time-delayed systems, which is presented in Chapter 7. Computationally tractable criteria for analyzing the stability of LPV *time-delayed* systems, including both delay-independent and delay-dependent stability, are derived using appropriately selected Lyapunov-Krasovskii functionals. Gridding techniques may be used to cast these tests as Linear Matrix Inequalities (LMI's). In cases when the system matrices depend affinely or quadratically on the parameter, gridding could be avoided.

## CHAPTER II

### MATHEMATICAL PRELIMINARIES

This chapter is self-contained. It introduces some basic mathematical concepts, tools, symbols and background material for use in subsequent chapters. It can be skipped at first reading and used as a reference in the subsequent chapters.

#### 2.1 *The Guardian Map*

Our results on robust stability of LTI parameter-dependent systems rely heavily on the concept of a guardian map for the set of Hurwitz matrices. A guardian map transforms a matrix stability problem to a non-singularity problem of an associated matrix. The most common guardian map is the one that involves the Kronecker sum of a matrix with itself. The definitions of the Kronecker product and Kronecker sum of two matrices may be found in several standard references (see for example [18]).

The following definition is taken from [7].

**Definition 2.1 (Guardian Map)** *Let an open set  $\mathcal{S} \subseteq \mathbb{R}^{n \times n}$  and  $\nu: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$  be a given mapping. Then  $\nu$  is said to guard the set  $\mathcal{S}$  if  $\nu(A) \neq 0$  for  $A \in \mathcal{S}$  and  $\nu(A) = 0$  for  $A \in \partial\mathcal{S}$ . The map  $\nu$  is called a guardian map for  $\mathcal{S}$ .*

##### 2.1.1 Guardian Map Induced by the Kronecker Sum

**Lemma 2.1 ([98])** *Let  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{m \times m}$ . Then  $\text{mspec}(A \oplus B) = \{\lambda_i + \mu_j : \lambda_i \in \text{mspec}(A), \mu_j \in \text{mspec}(B), i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, m\}$ .*

**Lemma 2.2 ([94])** *Given a matrix  $A \in \mathbb{R}^{n \times n}$ , define  $\bar{A} := A \oplus A$ . Assume that  $A$  is*



*Hurwitz. Then:*

(i)  $\bar{A}$  is Hurwitz

(ii)  $\det \bar{A} \neq 0$

*Proof.* Follows directly from the definition of  $\bar{A}$  and Lemma 2.1. ■

From Lemma 2.2 it is clear that the Kronecker sum induces the guardian map  $\nu_1 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$

$$\nu_1(A) := \det(A \oplus A) = \det \bar{A} \quad (13)$$

which guards the set  $\mathcal{A}$  of Hurwitz matrices [7].

### 2.1.2 Guardian Map Induced by the Bialternate Sum

For  $A, B \in \mathbb{R}^{n \times n}$  with elements  $a_{ij}$  and  $b_{ij}$ , the bialternate product of  $A$  and  $B$  is the matrix  $F = A \star B$  of dimension  $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ , with elements as follows [42, 54, 51]:

$$f_{\tilde{m}(n,p,q), \tilde{m}(n,r,s)} := \frac{1}{2} \left( \det \begin{bmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{bmatrix} + \det \begin{bmatrix} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{bmatrix} \right),$$

where the index function  $\tilde{m}$  is defined as:

$$\tilde{m}(n, i, j) := (j-1)n + i - \frac{1}{2}j(j+1). \quad (14)$$

According to this definition, it is clear that  $A \star B = B \star A$ . For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \quad (15)$$

then

$$A \star B = \frac{1}{2} \left[ \begin{array}{c} \begin{pmatrix} a_{22}b_{11} + a_{11}b_{22} \\ -a_{12}b_{21} - a_{21}b_{12} \end{pmatrix} \\ \begin{pmatrix} a_{11}b_{32} + a_{32}b_{11} \\ -a_{12}b_{31} - a_{31}b_{12} \end{pmatrix} \\ \begin{pmatrix} a_{21}b_{32} + a_{32}b_{21} \\ -a_{22}b_{31} - a_{31}b_{22} \end{pmatrix} \end{array} \begin{array}{c} \begin{pmatrix} a_{11}b_{23} + a_{23}b_{11} \\ -a_{21}b_{13} - a_{13}b_{21} \end{pmatrix} \\ \begin{pmatrix} a_{11}b_{33} + a_{33}b_{11} \\ -a_{13}b_{31} - a_{31}b_{13} \end{pmatrix} \\ \begin{pmatrix} a_{21}b_{33} + a_{33}b_{21} \\ -a_{23}b_{31} - a_{31}b_{23} \end{pmatrix} \end{array} \begin{array}{c} \begin{pmatrix} a_{12}b_{23} + a_{23}b_{12} \\ -a_{22}b_{13} - a_{13}b_{22} \end{pmatrix} \\ \begin{pmatrix} a_{12}b_{33} + a_{33}b_{12} \\ -a_{13}b_{32} - a_{32}b_{13} \end{pmatrix} \\ \begin{pmatrix} a_{22}b_{33} + a_{33}b_{22} \\ -a_{23}b_{32} - a_{32}b_{23} \end{pmatrix} \end{array} \right]. \quad (16)$$

The bialternate sum  $\tilde{A}$  of matrix  $A$  with itself is defined as [25, 42, 54]

$$\tilde{A} = A \star I_n + I_n \star A = 2A \star I_n. \quad (17)$$

If  $\tilde{a}_{ij}$  denotes the  $ij$ -th element of  $\tilde{A}$  then,

$$\tilde{a}_{\tilde{m}(n,p,q),\tilde{m}(n,r,s)} = \det \begin{bmatrix} a_{pr} & a_{ps} \\ \delta_{qr} & \delta_{qs} \end{bmatrix} + \det \begin{bmatrix} \delta_{pr} & \delta_{ps} \\ a_{qr} & a_{qs} \end{bmatrix} \quad (18)$$

where,  $\delta_{ij}$  is the Kronecker delta ( $\delta_{ij} = 1$ , if  $i = j$ ,  $\delta_{ij} = 0$ , if  $i \neq j$ ). Clearly, if  $A \in \mathbb{R}^{n \times n}$ , then  $\tilde{A} \in \mathbb{R}^{\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)}$ . For example, using (18), we have

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \tilde{A} = a_{11} + a_{22} \quad (19)$$

and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \tilde{A} = 2A \star I_n = \begin{bmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{bmatrix}. \quad (20)$$

From the definition of the bialternate sum of a matrix with itself, one has immediately that

$$\begin{aligned} \widetilde{\alpha A} &= \alpha \tilde{A} \\ \widetilde{A_0 + \rho A_g} &= \tilde{A}_0 + \rho \tilde{A}_g \end{aligned}$$

where  $A, A_0, A_g \in \mathbb{R}^{n \times n}$ , and  $\alpha, \rho \in \mathbb{R}$ .

**Theorem 2.1** ([42]) *Let  $A \in \mathbb{R}^{n \times n}$ . Then  $\text{mspec}(\tilde{A}) = \{\lambda_i(A) + \lambda_j(A), \quad i = 2, 3, \dots, n, j = 1, 2, \dots, i-1\}$ .*

The following corollary follows immediately from Theorem 2.1.

**Corollary 2.1** *Let  $A \in \mathbb{R}^{n \times n}$  be Hurwitz. Then:*

(i)  $\tilde{A}$  is Hurwitz.

(ii)  $\det \tilde{A} \neq 0$ .

**Remark 2.1** The determinant of the bialternate sum of a matrix with itself cannot be used as a guardian map of  $\mathcal{A}$ . To see this, let a matrix  $A \in \mathbb{R}^{n \times n}$  with only one eigenvalue at zero and all other eigenvalues in the open left half complex plane. In this case,  $A \in \partial\mathcal{A}$ , but  $\det \tilde{A} \neq 0$ . However, the map

$$\nu_2(A) = \det A \det \tilde{A} \quad (21)$$

is a guardian map which guards the set  $\mathcal{A}$ . First, it is easy to see that  $\nu_2(A) \neq 0$  if  $A \in \mathcal{A}$ . Moreover, if  $A \in \partial\mathcal{A}$ , some eigenvalues of the matrix  $A$  are on the  $j\omega$ -axis and all the others are in the open left half plane of  $\mathbb{C}$ . Let  $\mathcal{F}$  be the set of matrices in  $\partial\mathcal{A}$  with at most one eigenvalue at the origin

$$\mathcal{F} = \{A \in \partial\mathcal{A} : \lambda_i(A) = 0 \text{ and } \lambda_j(A) \neq 0 \text{ for all } j \neq i, \ i, j \in \mathcal{I}_n\}. \quad (22)$$

If  $A \in \mathcal{F}$  then  $\det A = 0$  and if  $A \in \partial\mathcal{A} \setminus \mathcal{F}$  then  $\det \tilde{A} = 0$ . In either case,  $\nu_2(A) = 0$ . Hence,  $\nu_2(A)$  is a guardian map for the set  $\mathcal{A}$  according to the Definition 2.1. Moreover,  $\nu_2(A)$  is easier to compute than  $\nu_1(A)$  since the dimension of  $\tilde{A}$  is  $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$  whereas that of  $\tilde{A}$  is  $n^2 \times n^2$ .

## 2.2 Computation of Adjoint Matrices

### 2.2.1 Single-Parameter Case

Given matrices  $A_0, A_g \in \mathbb{R}^{n \times n}$ , the following lemmas will be used to calculate the adjoint of the matrix  $A_0 + \rho A_g$ . Before that, the following definition is in order.

**Definition 2.2 ([85])** Let  $A, B \in \mathbb{R}^{n \times n}$ . Then  $\Gamma_n^i \det(A/B^i)$  is defined as the sum of determinants, in which the  $i$  rows of  $A$  are substituted by the corresponding rows of  $B$ .

For example, if  $A = \{a_{ij}\}$  and  $B = \{b_{ij}\} \in \mathbb{R}^{3 \times 3}$ , then

$$\begin{aligned}\Gamma_3^1 \det(A/B^1) &= \det \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \\ \Gamma_3^2 \det(A/B^2) &= \det \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} + \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}\end{aligned}$$

**Lemma 2.3** ([85]) *Let  $A, B \in \mathbb{R}^{n \times n}$ , then*

$$\det(A + B) = \det(A) + \det(B) + \sum_{i=1}^{n-1} \Gamma_n^i \det(A/B^i)$$

or

$$\det(A + B) = \sum_{i=0}^n \Gamma_n^i \det(A/B^i)$$

Using Lemma 2.3, one can use the following result to calculate the adjoint of the sum of two matrices.

**Lemma 2.4** ([84]) *Let  $A, B \in \mathbb{R}^{n \times n}$ , then*

$$\text{Adj}(A + B) = \text{Adj}(A) + \text{Adj}(B) + \sum_{i=1}^{n-2} \Gamma_{n-1}^i \text{Adj}(A/B^i)$$

where

$$\left[ \Gamma_{n-1}^i \text{Adj}(A/B^i) \right]_{kj} = (-1)^{k+j} \Gamma_{n-1}^i \det(A_{jk}/B_{jk}^i)$$

where  $\text{Adj}(\cdot)$  represents the adjoint matrix of  $(\cdot)$ , and  $(\cdot)_{jk}$  is a submatrix of  $(\cdot)$  in which the  $j$ -th row and the  $k$ -th column are eliminated.  $\left[ \Gamma_{n-1}^i \text{Adj}(A/B^i) \right]_{kj}$  is the  $kj$ -th element of the matrix  $\Gamma_{n-1}^i \text{Adj}(A/B^i)$ .

Lemma 2.4, together with the fact that  $\text{Adj}(\rho A) = \rho^{n-1} \text{Adj}(A)$  for  $A \in \mathbb{R}^{n \times n}$ , can be used to calculate the adjoint matrix of a parameter-dependent matrix as follows.

**Corollary 2.2** *Let  $A, B \in \mathbb{R}^{n \times n}$  and  $\rho \in \mathbb{R}$ , then*

$$\text{Adj}(A + \rho B) = \text{Adj}(A) + \rho^{n-1} \text{Adj}(B) + \sum_{i=1}^{n-2} \rho^i \Gamma_{n-1}^i \text{Adj}(A/B^i)$$

where the  $kj$ -th element of the matrix  $\Gamma_{n-1}^i \text{Adj}(A/B^i)$  is

$$\left[ \Gamma_{n-1}^i \text{Adj}(A/B^i) \right]_{kj} = (-1)^{k+j} \Gamma_{n-1}^i \det(A_{jk}/B_{jk}^i)$$

*Proof.* Denoting the  $kj$ -th element of  $\text{Adj}(A + \rho B)$  as  $\left[ \text{Adj}(A + \rho B) \right]_{kj}$ , then

$$\begin{aligned} \left[ \text{Adj}(A + \rho B) \right]_{kj} &= (-1)^{k+j} \det(A_{jk} + \rho B_{jk}) \\ &= (-1)^{k+j} \det(A_{jk}) + (-1)^{k+j} \det(\rho B_{jk}) + \sum_{i=1}^{n-2} (-1)^{k+j} \rho^i \Gamma_{n-1}^i \det(A_{jk}/B_{jk}^i) \end{aligned}$$

Since  $\text{Adj}(A) = \{(-1)^{k+j} \det(A_{jk})\}$  and  $\text{Adj}(B) = \{(-1)^{k+j} \det(B_{jk})\}$ , one has

$$\begin{aligned} \text{Adj}(A + \rho B) &= \left\{ \left[ \text{Adj}(A + \rho B) \right]_{kj} \right\} \\ &= \text{Adj}(A) + \rho^{n-1} \text{Adj}(B) + \sum_{i=1}^{n-2} \rho^i \Gamma_{n-1}^i \text{Adj}(A/B^i) \end{aligned}$$

and the proof is complete. ■

### 2.2.2 Two-Parameter Case

Given the matrices  $A_0, A_1, A_2 \in \mathbb{R}^{n \times n}$ , the following results will be used to calculate the adjoint matrix  $\text{Adj}(A_0 + \rho_1 A_1 + \rho_2 A_2)$ .

**Definition 2.3 ([86])** *Let  $A, B$  and  $C \in \mathbb{R}^{n \times n}$ . Then,  $\Gamma_n^i \det(A/B^k C^{i-k})$  is defined as the sum of determinants, in which the  $i$  rows of  $A$  are substituted by the corresponding  $k$  rows of  $B$  and  $i - k$  rows of  $C$ , where  $k \leq i$ .*

For example, let  $a_j$ ,  $b_j$  and  $c_j$  denote the  $j$ th row of  $A$ ,  $B$  and  $C \in \mathbb{R}^{4 \times 4}$  respectively. Then

$$\begin{aligned} \Gamma_4^2 \det(A/B^1 C^1) &= \det \begin{bmatrix} c_1 \\ b_2 \\ a_3 \\ a_4 \end{bmatrix} + \det \begin{bmatrix} c_1 \\ a_2 \\ b_3 \\ a_4 \end{bmatrix} + \det \begin{bmatrix} c_1 \\ a_2 \\ a_3 \\ b_4 \end{bmatrix} + \det \begin{bmatrix} a_1 \\ c_2 \\ b_3 \\ a_4 \end{bmatrix} + \det \begin{bmatrix} a_1 \\ c_2 \\ a_3 \\ b_4 \end{bmatrix} + \det \begin{bmatrix} b_1 \\ c_2 \\ a_3 \\ a_4 \end{bmatrix} \\ &+ \det \begin{bmatrix} a_1 \\ a_2 \\ c_3 \\ b_4 \end{bmatrix} + \det \begin{bmatrix} b_1 \\ a_2 \\ c_3 \\ a_4 \end{bmatrix} + \det \begin{bmatrix} a_1 \\ b_2 \\ c_3 \\ a_4 \end{bmatrix} + \det \begin{bmatrix} b_1 \\ a_2 \\ a_3 \\ c_4 \end{bmatrix} + \det \begin{bmatrix} a_1 \\ b_2 \\ a_3 \\ c_4 \end{bmatrix} + \det \begin{bmatrix} a_1 \\ a_2 \\ b_3 \\ c_4 \end{bmatrix} \end{aligned}$$

The following properties are obvious [86]

$$\Gamma_n^i \det(A/B^k C^{i-k}) = \Gamma_n^i \det(A/C^{i-k} B^k) \quad (23)$$

$$\begin{aligned} \Gamma_n^i \det(A/B^k C^{i-k}) &= \Gamma_n^{n-k} \det(B/A^{n-i} C^{i-k}) \\ &= \Gamma_n^{n-i+k} \det(C/A^{n-i} B^k) \end{aligned} \quad (24)$$

The following theorem is a generalization of Lemma 2.3.

**Theorem 2.2** ([86]) *Let  $A, B \in \mathbb{R}^{n \times n}$ , then*

$$\begin{aligned} \det(A + B + C) &= \det(A) + \det(B) + \det(C) + \sum_{i=1}^{n-1} \Gamma_n^i \det(A/B^i) \\ &+ \sum_{i=1}^{n-1} \Gamma_n^i \det(A/C^i) + \sum_{i=1}^{n-1} \Gamma_n^i \det(B/C^i) \\ &+ \sum_{i=2}^{n-1} \sum_{k=1}^{i-1} \Gamma_n^i \det(A/B^k C^{i-k}) \end{aligned} \quad (25)$$

$$\det(A + B + C) = \sum_{i=0}^n \sum_{k=0}^i \Gamma_n^i \det(A/B^k C^{i-k}) \quad (26)$$

According to the definition of  $\Gamma_n^i \det(A/B^k C^{i-k})$ , it follows that

$$\Gamma_n^i \det\left(A/(\rho_1 B)^k (\rho_2 C)^{i-k}\right) = \rho_1^k \rho_2^{i-k} \Gamma_n^i \det(A/B^k C^{i-k}) \quad (27)$$

The following corollary can be applied to calculate the adjoint matrix of a two-parameter dependent matrix.

**Corollary 2.3** *Let  $A, B \in \mathbb{R}^{n \times n}$  and  $\rho_1, \rho_2 \in \mathbb{R}$ . Then  $\text{Adj}(A + \rho_1 B + \rho_2 C)$  can be calculated by the following formula*

$$\left[ \text{Adj}(A + \rho_1 B + \rho_2 C) \right]_{kj} = (-1)^{k+j} \sum_{i=0}^{n-1} \sum_{k=0}^i \rho_1^k \rho_2^{i-k} \Gamma_{n-1}^i \det(A_{jk}/B_{jk}^k C_{jk}^{i-k}) \quad (28)$$

where  $\left[ \text{Adj}(A + \rho_1 B + \rho_2 C) \right]_{kj}$  is the  $kj$ -th element of the matrix  $\text{Adj}(A + \rho_1 B + \rho_2 C)$ , and  $A_{jk}, B_{jk}, C_{jk}$  are the submatrices of  $A, B, C$  with the  $j$ th row and  $k$ th column eliminated.

*Proof.* Notice that for the  $kj$ -th element of the  $\text{Adj}(A + \rho_1 B + \rho_2 C)$ , we have that

$$\left[ \text{Adj}(A + \rho_1 B + \rho_2 C) \right]_{kj} = (-1)^{k+j} \det(A_{jk} + \rho_1 B_{jk} + \rho_2 C_{jk}) \quad (29)$$

Since  $A_{jk} + \rho_1 B_{jk} + \rho_2 C_{jk} \in \mathbb{R}^{(n-1) \times (n-1)}$  and using Theorem 2.2 and property (27), it follows that

$$\begin{aligned} \det(A_{jk} + \rho_1 B_{jk} + \rho_2 C_{jk}) &= \sum_{i=0}^{n-1} \sum_{k=0}^i \Gamma_{n-1}^i \det(A_{jk}/(\rho_1 B_{jk})^k (\rho_2 C_{jk})^{i-k}) \\ &= \sum_{i=0}^{n-1} \sum_{k=0}^i \rho_1^k \rho_2^{i-k} \Gamma_{n-1}^i \det(A_{jk}/B_{jk}^k C_{jk}^{i-k}) \end{aligned} \quad (30)$$

Substituting  $\det(A_{jk} + \rho_1 B_{jk} + \rho_2 C_{jk})$  with (30) into the right side of (29), (28) follows.  $\blacksquare$

### 2.2.3 Multi-Parameter Case

The algorithm in this section for calculating the adjoint matrix of a multi-parameter dependent matrix is based on the one for a single- or two-parameter dependent matrix. Given the matrices  $A_0, A_1, \dots, A_m \in \mathbb{R}^{n \times n}$  and  $\rho = (\rho_1, \dots, \rho_m)^T \in \mathbb{R}^m$ , let

$$A(\rho) = A_0 + \rho_1 A_1 + \dots + \rho_m A_m$$

Notice that for the  $kj$ -th element of the  $\text{Adj}(A(\rho))$ , we have that

$$\left[ \text{Adj}(A_0 + \rho_1 A_1 + \dots + \rho_m A_m) \right]_{kj} = (-1)^{k+j} \det(A_{0,jk} + \rho_1 A_{1,jk} + \dots + \rho_m A_{m,jk}) \quad (31)$$

where  $\left[ \text{Adj}(A_0 + \rho_1 A_1 + \dots + \rho_m A_m) \right]_{kj}$  is the  $kj$ -th element of the matrix  $\text{Adj}(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)$ , and  $A_{0,jk}, A_{1,jk}, \dots, A_{m,jk}$  are the submatrices of  $A_0, A_1, \dots, A_m$  with the

$j$ th row and  $k$ th column eliminated. Using Lemma 2.3 and since  $A_{0,jk}, A_{1,jk}, \dots, A_{m,jk} \in \mathbb{R}^{(n-1) \times (n-1)}$ , one has

$$\begin{aligned}
& \det(A_{0,jk} + \rho_1 A_{1,jk} + \dots + \rho_m A_{m,jk}) \\
&= \det\left((A_{0,jk} + \rho_1 A_{1,jk} + \dots + \rho_{m-1} A_{m-1,jk}) + \rho_m A_{m,jk}\right) \\
&= \det(A_{0,jk} + \rho_1 A_{1,jk} + \dots + \rho_{m-1} A_{m-1,jk}) + \det(\rho_m A_{m,jk}) \\
&\quad + \sum_{i=1}^{n-2} \Gamma_{n-1}^i \det\left((A_{0,jk} + \rho_1 A_{1,jk} + \dots + \rho_{m-1} A_{m-1,jk}) / (\rho_m A_{m,jk})^i\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \det(A_{0,jk} + \rho_1 A_{1,jk} + \dots + \rho_m A_{m,jk}) \\
&= \det(A_{0,jk} + \rho_1 A_{1,jk} + \dots + \rho_{m-1} A_{m-1,jk}) + \det(\rho_m A_{m,jk}) \\
&\quad + \sum_{i=1}^{n-2} \Gamma_{n-1}^i \rho_m^i \det\left((A_{0,jk} + \rho_1 A_{1,jk} + \dots + \rho_{m-1} A_{m-1,jk}) / (A_{m,jk})^i\right) \quad (32)
\end{aligned}$$

In Eq. (32), the left-hand side is  $m$ -parameter dependent, while the terms  $\det(A_{0,jk} + \rho_1 A_{1,jk} + \dots + \rho_{m-1} A_{m-1,jk})$  and  $\det((A_{0,jk} + \rho_1 A_{1,jk} + \dots + \rho_{m-1} A_{m-1,jk}) / (A_{m,jk})^i)$  in the right-hand side of Eq. (32) are  $(m-1)$ -parameter dependent. Thus, Eq. (32) shows that the determinant of an affinely  $m$ -parameter dependent matrix can be expressed as the sum of a finite number of determinants of  $(m-1)$ -parameter dependent matrices. A recursive calculation using Eq. (32) shows that, the determinant of a  $m$ -parameter affinely dependent matrix can be expressed as the sum of a finite number of determinants of two-parameter dependent matrices, which can be computed using the algorithm introduced in Section 2.2.2.

After the calculation of  $\det((A(\rho))_{jk})$ , with Eq. (31), one can compute the adjoint matrix of a  $m$ -parameter affinely dependent matrix.

**Remark 2.2** From the procedure of computing the adjoint matrix of a  $m$ -parameter affinely dependent matrix as introduced above, the increase of parameter number  $m$  lead to great growth in computation.



### 2.3 Positive Definite Functionals

Let us denote by  $\mathcal{C}_\tau$  the set of continuous functions defined over the interval  $[-\tau, 0]$  and let  $V : \mathbb{R}_+ \times \mathcal{C}_\tau \rightarrow \mathbb{R}_+$  be a continuous functional such that  $V(t, 0) = 0$ . Let also  $\Omega$  denote the class of scalar, nondecreasing continuous functions  $\alpha$  such that  $\alpha(r) > 0$  for  $r > 0$  and  $\alpha(0) = 0$ . The functional  $V(t, \psi)$  is called *positive definite (negative definite)* if there exist a function  $\alpha \in \Omega$  such that  $V(t, \psi) \geq \alpha(|\psi(0)|)^1$  (respectively,  $V(t, \psi) \leq -\alpha(|\psi(0)|)$ ) for all  $t \in \mathbb{R}$  and  $\psi \in \mathcal{C}_\tau$ . It is said to have an *infinitesimal upper bound* if  $|V(t, \psi)| \leq \alpha(\sup_t |\psi(t)|)$ .

The following lemma is useful for recognizing positive definite functionals, as the ones used in Chapter 7. In the following,  $x_t \in \mathcal{C}_\tau$  denotes the function with domain  $[-\tau, 0]$  that coincides with  $x$  in the interval  $[t - \tau, t]$  i.e.,  $x_t : [-\tau, 0] \rightarrow \mathbb{R}^n$  such that  $x_t(\theta) = x(t + \theta)$  for  $\theta \in [-\tau, 0]$ . In the sequel,  $|x|$  will denote the euclidean norm of a vector  $x \in \mathbb{R}^n$ . Moreover, given a matrix  $A(\gamma) \in \mathbb{R}^{n \times m}$ , depending continuously on a parameter  $\gamma$  that belongs to a compact set  $\Gamma$ , we denote

$$\|A\|_{\infty, \Gamma} = \left[ \max_{\gamma \in \Gamma} \bar{\sigma}^2(A(\gamma)) \right]^{1/2} \quad (33)$$

where  $\bar{\sigma}(A)$  is the maximum singular value of  $A$ .  $\|A\|_{\infty, \Gamma}$  is always well-defined since  $\Gamma$  is compact and the singular value is a continuous function of the elements of a matrix [38].

**Lemma 2.5** *Consider the functional  $V : \mathbb{R}_+ \times \mathcal{C}_{2\tau} \rightarrow \mathbb{R}_+$  given by*

$$\begin{aligned} V(t, x_t) &= x^T(t)Px(t) + \int_{-\tau}^0 \int_{\beta}^0 [A(\gamma(t+\alpha))x(t+\alpha)]^T P_1 [A(\gamma(t+\alpha))x(t+\alpha)] \, d\alpha \, d\beta \\ &+ \int_{-\tau}^0 \int_{\beta}^0 [A_d(\gamma(t+\alpha))x(t+\alpha-\tau)]^T P_2 [A_d(\gamma(t+\alpha))x(t+\alpha-\tau)] \, d\alpha \, d\beta \\ &+ \int_{-\tau}^0 \int_{\beta}^0 [A(\gamma(t+\alpha))x(t+\alpha) + A_d(\gamma(t+\alpha))x(t+\alpha-\tau)]^T Y [A(\gamma(t+\alpha))x(t+\alpha) \\ &+ A_d(\gamma(t+\alpha))x(t+\alpha-\tau)] \, d\alpha \, d\beta + \int_{-\tau}^0 x^T(t+\alpha)Qx(t+\alpha) \, d\alpha \end{aligned} \quad (34)$$

where  $P, P_1, P_2, Q, Y$  are constant positive-definite matrices,  $A(\gamma)$  and  $A_d(\gamma)$  are matrices

---

<sup>1</sup>The norm used here need not be the usual Euclidian norm. If the state space is of finite dimension  $n$ , stability and its type are independent of the choice of norm (all norms are equivalent), although a particular choice of norm may make the analysis easier [68].

that depend continuously on the parameter  $\gamma \in \Gamma$ , with  $\Gamma$  compact. Then for every  $\tau > 0$  this  $V$  is a positive definite functional and it has an infinitesimal upper bound.

*Proof.* To show that  $V$  is a positive definite functional, let  $x_t(\theta) = x(t+\theta)$  and notice that

$$V(t, x_t) > x_t^T(0) P x_t(0) \geq \lambda_{\min}(P) |x_t(0)|^2$$

where  $\lambda_{\min}(P)$  denotes the minimum eigenvalue of  $P$ . In order to show that  $V$  has an infinitesimal upper bound, first notice that

$$V_1 = x_t^T(0) P x_t(0) \leq \lambda_{\max}(P) |x_t(0)|^2$$

Also

$$\begin{aligned} V_2 &= \int_{-\tau}^0 \int_{\beta}^0 [A(\gamma(t+\alpha))x(t+\alpha)]^T P_1 [A(\gamma(t+\alpha))x(t+\alpha)] \, d\alpha \, d\beta \\ &\leq \int_{-\tau}^0 \int_{\beta}^0 \lambda_{\max}(P_1) |A(\gamma)x(t+\alpha)|^2 \, d\alpha \, d\beta \\ &\leq \int_{-\tau}^0 \int_{\beta}^0 \lambda_{\max}(P_1) \|A\|_{\infty, \Gamma}^2 \max_{\theta} |x(t+\theta)|^2 \, d\alpha \, d\beta \\ &= \lambda_{\max}(P_1) \|A\|_{\infty, \Gamma}^2 \max_{\theta} |x(t+\theta)|^2 \int_{-\tau}^0 \int_{\beta}^0 \, d\alpha \, d\beta \end{aligned}$$

Let now  $\lambda_2 = \lambda_{\max}(P_1) \|A\|_{\infty, \Gamma}^2 (\frac{1}{2}\tau^2)$ . Then

$$V_2 \leq \lambda_2 \max_{\theta \in [-\tau, 0]} |x(t+\theta)|^2 = \lambda_2 \max_{\theta \in [-\tau, 0]} |x_t(\theta)|^2$$

Moreover,

$$\begin{aligned} V_3 &= \int_{-\tau}^0 \int_{\beta}^0 [A_d(\gamma(t+\alpha))x(t+\alpha-\tau)]^T P_2 [A_d(\gamma(t+\alpha))x(t+\alpha-\tau)] \, d\alpha \, d\beta \\ &\leq \int_{-\tau}^0 \int_{\beta}^0 \lambda_{\max}(P_2) |A_d(\gamma)x(t+\alpha-\tau)|^2 \, d\alpha \, d\beta \\ &\leq \int_{-\tau}^0 \int_{\beta}^0 \lambda_{\max}(P_2) \|A_d\|_{\infty, \Gamma}^2 \max_{\theta} |x(t+\theta)|^2 \, d\alpha \, d\beta \\ &= \lambda_{\max}(P_2) \|A_d\|_{\infty, \Gamma}^2 \max_{\theta} |x(t+\theta)|^2 \int_{-\tau}^0 \int_{\beta}^0 \, d\alpha \, d\beta \end{aligned}$$

Let now  $\lambda_3 = \lambda_{\max}(P_2) \|A_d\|_{\infty, \Gamma}^2 (\frac{1}{2}\tau^2)$ . Then

$$V_3 \leq \lambda_3 \max_{\theta \in [-2\tau, 0]} |x(t+\theta)|^2$$

Similarly,

$$\begin{aligned}
V_4 &= \int_{-\tau}^0 x^T(t+\alpha)Qx(t+\alpha) \, d\alpha \\
&\leq \lambda_{\max}(Q) \int_{-\tau}^0 \max_{\theta \in [-\tau, 0]} |x(t+\theta)|^2 \, d\alpha \\
&= \tau \lambda_{\max}(Q) \max_{\theta \in [-\tau, 0]} |x(t+\theta)|^2 = \tau \lambda_{\max}(Q) \max_{\theta \in [-\tau, 0]} |x_t(\theta)|^2
\end{aligned}$$

Finally,

$$\begin{aligned}
V_5 &= \int_{-\tau}^0 \int_{\beta}^0 [A(\gamma(t+\alpha))x(t+\alpha) + A_d(\gamma(t+\alpha))x(t+\alpha-\tau)]^T Y \\
&\quad \times [A(\gamma(t+\alpha))x(t+\alpha) + A_d(\gamma(t+\alpha))x(t+\alpha-\tau)] \, d\alpha \, d\beta \\
&\leq \lambda_{\max}(Y) \int_{-\tau}^0 \int_{\beta}^0 |A(\gamma(t+\alpha))x(t+\alpha) + A_d(\gamma(t+\alpha))x(t+\alpha-\tau)|^2 \, d\alpha \, d\beta \\
&\leq \lambda_{\max}(Y) \int_{-\tau}^0 \int_{\beta}^0 [2|A(\gamma(t+\alpha))x(t+\alpha)|^2 + 2|A_d(\gamma(t+\alpha))x(t+\alpha-\tau)|^2] \, d\alpha \, d\beta \\
&\leq 2\lambda_{\max}(Y) \int_{-\tau}^0 \int_{\beta}^0 [\|A\|_{\infty, \Gamma}^2 \max_{\theta} |x(t+\theta)|^2 + \|A_d\|_{\infty, \Gamma}^2 \max_{\theta} |x(t+\theta-\tau)|^2] \, d\alpha \, d\beta \\
&\leq \tau^2 \lambda_{\max}(Y) [\|A\|_{\infty, \Gamma}^2 \max_{\theta} |x(t+\theta)|^2 + \|A_d\|_{\infty, \Gamma}^2 \max_{\theta} |x(t+\theta-\tau)|^2]
\end{aligned}$$

From the previous inequalities for  $V_i$  ( $i = 1, \dots, 5$ ) it follows that there exist constants  $c_0, c_1, c_2, c_3$  such that

$$\begin{aligned}
V(t, x_t) &\leq c_0 |x_t(0)|^2 + c_1 \max_{\theta \in [-\tau, 0]} |x_t(\theta)|^2 + c_2 \max_{\theta \in [-2\tau, 0]} |x_t(\theta)|^2 \\
&\leq c_3 \max_{\theta \in [-2\tau, 0]} |x_t(\theta)|^2
\end{aligned}$$

Therefore,  $V$  is a positive definite functional with an infinitesimal upper bound. ■

The following corollary follows immediately.

**Corollary 2.4** *Consider the functional  $V : \mathbb{R}_+ \times \mathcal{C}_{2\tau} \rightarrow \mathbb{R}_+$*

$$\begin{aligned}
V(t, x_t) &= x^T(t)Px(t) + \int_{-\tau}^0 \int_{\beta}^0 [A(\gamma(t+\alpha))x(t+\alpha)]^T P_1 [A(\gamma(t+\alpha))x(t+\alpha)] \, d\alpha \, d\beta \\
&\quad + \int_{-\tau}^0 \int_{\beta}^0 [A_d(\gamma(t+\alpha))x(t+\alpha-\tau)]^T P_2 [A_d(\gamma(t+\alpha))x(t+\alpha-\tau)] \, d\alpha \, d\beta
\end{aligned}$$

where  $P, P_1$  and  $P_2$  are constant, positive-definite matrices and  $A(\gamma)$  and  $A_d(\gamma)$  are matrices that depend continuously on the parameter  $\gamma \in \Gamma$ , with  $\Gamma$  compact. Then for every  $\tau > 0$ ,  $V$  is positive definite with an infinitesimal upper bound.

The following lemma also holds.

**Lemma 2.6** *Let  $\Gamma$  be a compact interval of the real line. Consider the continuous functional*

*$V : \mathbb{R}_+ \times \mathcal{C}_\tau \rightarrow \mathbb{R}_+$  defined by*

$$V(t, x_t) = x^T(t)P(\gamma(t))x(t) + \int_{-\tau}^0 x^T(t+\theta)Q(\gamma(t+\theta))x(t+\theta) \, d\theta$$

*where  $\gamma(t) \in \Gamma$  for all  $t \geq 0$  and  $P(\gamma) > 0$  and  $Q(\gamma) > 0$  for all  $\gamma \in \Gamma$ . Then  $V$  is a positive definite functional with an infinitesimal upper bound.*

*Proof.* The proof is similar to the one for Lemma 2.5 and will not be repeated here. ■

## CHAPTER III

### ROBUST STABILITY OF LTI PARAMETER-DEPENDENT SYSTEMS

#### 3.1 *Introduction*

The research work in this chapter is motivated by the need of stability analysis of active magnetic bearings (AMB) supporting flexible gyroscopic mechanical systems as outlined in Section 1.1. Such a system can be described as a slow LPV model as follows

$$\dot{X} = (A_0 + \rho A_g)X \quad (35)$$

where  $\rho \in \mathbb{R}$  and the matrix  $A_0$  is usually assumed to be Hurwitz. The existing results [7, 24, 64] give the maximal continuous interval including  $\rho = 0$  such that  $A_0 + \rho A_g$  is Hurwitz when the parameter  $\rho$  takes a value inside this interval. Compared to existing results, which provide only sufficient stability conditions for LPV or LTIPD systems, we give next necessary and sufficient stability conditions for those systems. The whole stability domain of the parameter  $\rho$  may be composed of several disconnected intervals in  $\mathbb{R}$ . We extend the existing results and give a method to calculate the whole stability domain of  $\rho$ . The second contribution is the improvement of this algorithm using the guardian map induced by the bialternate sum. The new algorithm has the benefit of requiring fewer computations compared to the one which uses a guardian map induced by the Kronecker sum. The third contribution is the generalization of this algorithm to the case when the parameter  $\rho$  is a vector, i.e, to the multi-parameter dependent LPV systems

$$\dot{X} = (A_0 + \rho_1 A_{g,1} + \rho_2 A_{g,2} + \rho_3 A_{g,3} + \dots + \rho_k A_{g,k})X \quad (36)$$

The results in this chapter rely heavily on the concept of a guardian map for the set of Hurwitz matrices, which was presented in Chapter 2.

The current chapter has seven sections and is organized as follows: Section 3.2 defines two operators for a given square matrix. These definitions are used in the subsequent sections. Section 3.3 introduces two methods for computing the maximal open stability interval on  $\mathbb{R}$  which includes zero, such that the single parameter-dependent system matrix will be Hurwitz if the parameter is within this interval. This result is the same as the one in [24] and is included here for completeness, albeit with an alternate proof. The methods in Section 3.3 have the limitation that the system matrix must be Hurwitz when the parameter is zero. The guardian map induced by the Kronecker sum and the map induced by the bialternate sum are exploited to derive such a maximal open stability interval that includes the origin. Section 3.4 extends the results of Section 3.3 and gives two algorithms for computing the complete stability domain for a single parameter-dependent system matrix. This domain may be an open interval or a union of several open intervals. When the parameter is zero, the system matrix is not required to be Hurwitz in order to apply these two algorithms. Section 3.5 generalizes these results to multi-parameter dependent LTI systems. Section 3.6 gives some numerical examples and Section 3.7 presents the conclusions.

### ***3.2 Mathematical Preliminaries for Robust Stability of LTI Parameter Dependent Systems***

The following definition will be used extensively in the subsequent sections.

**Definition 3.1** *Given a matrix  $M \in \mathbb{R}^{n \times n}$ , let  $\tilde{\lambda}_i(M)$ ,  $i = 1, \dots, p$  denote the real, distinct, non-zero eigenvalues of  $M$  and define  $\tilde{\lambda}_0(M) = 0$ . If  $p = 0$ , let  $\mathcal{N}(M) = (-\infty, +\infty)$ , otherwise define the open interval  $\mathcal{N}(M)$  as follows:*

$$\mathcal{N}(M) := \left( -\frac{1}{\max_{i \in \mathcal{I}_p^0} \tilde{\lambda}_i(M)}, -\frac{1}{\min_{i \in \mathcal{I}_p^0} \tilde{\lambda}_i(M)} \right) \quad (37)$$

where,

$$\begin{aligned} -\frac{1}{\max_{i \in \mathcal{I}_p^0} \tilde{\lambda}_i(M)} &= -\infty, \quad \text{if} \quad \max_{i \in \mathcal{I}_p^0} \tilde{\lambda}_i(M) = 0, \\ -\frac{1}{\min_{i \in \mathcal{I}_p^0} \tilde{\lambda}_i(M)} &= +\infty, \quad \text{if} \quad \min_{i \in \mathcal{I}_p^0} \tilde{\lambda}_i(M) = 0. \end{aligned} \quad (38)$$

The following corollary is a direct consequence of Definition 3.1.

**Corollary 3.1** *For any  $M \in \mathbb{R}^{n \times n}$ ,*

- (i)  $0 \in \mathcal{N}(M)$
- (ii)  $\det(I + \rho M) \neq 0$ , for  $\rho \in \mathcal{N}(M)$

According to Corollary 3.1, for any non-singular, and dimensionally compatible matrix  $P$ , the matrices  $P + \rho PM$  and  $P + \rho MP$  are non-singular for all  $\rho \in \mathcal{N}(M)$ . By the definition of  $\mathcal{N}(M)$ , Corollary 3.1 gives an open interval in  $\mathbb{R}$ , which includes zero, and is the maximal interval such that the matrix  $I + \rho M$  is non-singular. To find all the possible open intervals in  $\mathbb{R}$  in addition to  $\mathcal{N}(M)$  such that the matrix  $I + \rho M$  is non-singular, we must first find all the possible values for  $\rho$  such that the matrix  $I + \rho M$  is singular.

**Definition 3.2** *Given  $M \in \mathbb{R}^{n \times n}$ , let  $\tilde{\lambda}_i(M)$ ,  $i = 1, \dots, p$  denote the real, distinct, non-zero eigenvalues of  $M$ . Let  $r_0 = -\infty$ ,  $r_i = -1/\tilde{\lambda}_i(M)$ ,  $i = 1, 2, \dots, p$  and  $r_{p+1} = +\infty$  and define the ordered set (after, perhaps, a relabelling of the indices)  $\mathcal{B}(M) := \{r_0, r_1, r_2, \dots, r_p, r_{p+1}\}$  such that  $r_i < r_{i+1}$ .*

**Remark 3.1** From the definition of  $\mathcal{B}(M)$ , it follows that,  $\det(I + rM) = 0$  if and only if  $r \in \mathcal{B}(M)$ .

### 3.3 Maximal Stability Domain of Single LTIPD Systems

#### 3.3.1 Existing Results

Saydy *et al.* [65] and Barmish [7] derived stability conditions for a family of  $n \times n$  parameter-dependent matrices given by  $A(\rho) = \sum_{i=0}^l \rho^i A_i$ . Their result tests whether  $A(\rho)$  is robustly stable for all  $\rho \in [0, 1]$ . Reference [63] provides an interval which guarantees robust stability for single- and multi-parameter dependent LTI systems. However, this interval is derived from sufficient conditions and hence it is not the maximal robust stability interval. Fu and Barmish [24] presented a method to synthesize the maximal stability interval that contains the origin for single parameter-dependent LTI systems. Next, we re-state the theorem in [24] giving an alternate proof. In the next section, Theorem 3.1 is extended so as to reduce the computations involved through the use of the bialternate sum of matrices.

**Theorem 3.1** *Given an open interval  $\Omega$  in  $\mathbb{R}$ , and  $A_0, A_g \in \mathbb{R}^{n \times n}$ , the following two statements are equivalent:*

- (i)  $0 \in \Omega$ , and  $A(\rho) := A_0 + \rho A_g$  is Hurwitz for all  $\rho \in \Omega$
- (ii)  $A_0$  is Hurwitz and  $0 \in \Omega \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$

The proof can be found in [24]. Here, we give an alternate proof of this result.

*Proof.* If  $A(\rho) = A_0 + \rho A_g$  then we can write

$$\begin{aligned} \bar{A}(\rho) &:= A(\rho) \oplus A(\rho) = A(\rho) \otimes I_n + I_n \otimes A(\rho) \\ &= (A_0 \otimes I_n + I_n \otimes A_0) + \rho(A_g \otimes I_n + I_n \otimes A_g) \\ &= \bar{A}_0 + \rho \bar{A}_g \end{aligned}$$

(i)  $\Rightarrow$  (ii): If  $A(\rho)$  is Hurwitz for all  $\rho \in \Omega$  and  $0 \in \Omega$ , then  $A_0$  is Hurwitz. Then by Lemma 2.2 it follows that  $\det \bar{A}_0 \neq 0$ , and  $\bar{A}_0^{-1}$  exists. Furthermore, since  $A(\rho)$  is Hurwitz for all  $\rho \in \Omega$  then also from Lemma 2.2,  $\det \bar{A}(\rho) \neq 0$  for all  $\rho \in \Omega$ . Therefore,

$$\begin{aligned} 0 \neq \det(\bar{A}_0 + \rho \bar{A}_g) &= \det[\bar{A}_0(I + \rho \bar{A}_0^{-1} \bar{A}_g)] \\ &= \det \bar{A}_0 \det(I + \rho \bar{A}_0^{-1} \bar{A}_g), \quad \forall \rho \in \Omega. \end{aligned}$$



Hence,  $0 \neq \det(I + \rho \bar{A}_0^{-1} \bar{A}_g)$  for all  $\rho \in \Omega$ . Using the Schur Decomposition Lemma [38], there exists a unitary matrix  $U \in \mathbb{C}^{n^2 \times n^2}$  and an upper triangular matrix  $T \in \mathbb{C}^{n^2 \times n^2}$  such that  $\bar{A}_0^{-1} \bar{A}_g = UTU^*$ . The diagonal elements of the matrix  $T$  are the eigenvalues of the matrix  $\bar{A}_0^{-1} \bar{A}_g$ . Thus, we have that  $\det(I + \rho T) \neq 0$  for all  $\rho \in \Omega$ . This, in turn, implies that

$$\rho \lambda_i(\bar{A}_0^{-1} \bar{A}_g) \neq -1, \quad \forall i \in \mathcal{I}_n, \forall \rho \in \Omega. \quad (39)$$

If  $\bar{A}_0^{-1} \bar{A}_g$  has no real eigenvalues or if the only real eigenvalues lie at the origin then  $\mathcal{N}(\bar{A}_0^{-1} \bar{A}_g) = (-\infty, \infty)$  and trivially  $\Omega \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$ . If  $\bar{A}_0^{-1} \bar{A}_g$  has some non-zero real eigenvalues, then the largest interval which includes  $\rho = 0$  such that  $\rho \lambda_i(\bar{A}_0^{-1} \bar{A}_g) \neq -1$ ,  $i = 1, 2, \dots, n$  is given by the definition of  $\mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$ . It follows from (39) that  $\Omega \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$ .

(ii)  $\Rightarrow$  (i): The proof follows by contradiction. To this end, assume  $0 \in \Omega$  and suppose  $A(\rho)$  is not Hurwitz for all  $\rho \in \Omega$ . Then, there exists a  $\rho_1 \in \Omega$  such that  $\text{Re}[\lambda_k(A(\rho_1))] \geq 0$  for some  $k \in \mathcal{I}_n$ . If  $\rho_1 = 0$ , the proof is complete since  $A_0$  is Hurwitz. Consequently, and without loss of generality, we may assume that  $\rho_1 > 0$  (the case for  $\rho_1 < 0$  being identical). Because  $\text{Re}[\lambda_i(A_0)] < 0$  for every  $i \in \mathcal{I}_n$  and the eigenvalues of  $A(\rho)$  change continuously with  $\rho$  (see [38], Appendix D), there exists  $\rho_2 \in (0, \rho_1] \subseteq \Omega$  such that  $\text{Re}[\lambda_k(A(\rho_2))] = 0$  for some  $k \in \mathcal{I}_n$ . There are two possibilities:

1.  $\lambda_k(A(\rho_2)) = 0$ . Then by Lemma 2.1, there exists  $m \in \mathcal{I}_{n^2}$  such that  $\lambda_m(\bar{A}(\rho_2)) = \lambda_k(A(\rho_2)) + \lambda_k(A(\rho_2)) = 0$ .
2.  $\lambda_k(A(\rho_2)) = j\omega$  and  $\omega \neq 0$ . Since  $A(\rho_2) \in \mathbb{R}^{n \times n}$ , there exists  $k' \in \mathcal{I}_n$  such that  $\lambda_{k'}(A(\rho_2)) = -j\omega$  and hence by Lemma 2.1, there exists  $m \in \mathcal{I}_{n^2}$  such that  $\lambda_m(\bar{A}(\rho_2)) = \lambda_k(A(\rho_2)) + \lambda_{k'}(A(\rho_2)) = 0$ .

Consequently, in either case, there exists  $m \in \mathcal{I}_{n^2}$  such that  $\lambda_m(\bar{A}(\rho_2)) = 0$  with  $\rho_2 \in \Omega$  and  $\det \bar{A}(\rho_2) = 0$ . However, since  $A_0$  is Hurwitz,  $\bar{A}_0^{-1}$  exists (by Lemma 2.2) and we can

write:

$$\begin{aligned}
0 &= \det \bar{A}(\rho_2) = \det(\bar{A}_0 + \rho_2 \bar{A}_g) \\
&= \det[\bar{A}_0(I + \rho_2 \bar{A}_0^{-1} \bar{A}_g)] \\
&= \det \bar{A}_0 \det(I + \rho_2 \bar{A}_0^{-1} \bar{A}_g) .
\end{aligned}$$

Since  $\det \bar{A}_0 \neq 0$  ( $A_0$  is Hurwitz), it follows necessarily that  $\det(I + \rho_2 \bar{A}_0^{-1} \bar{A}_g) = 0$ . This contradicts the fact that  $\rho_2 \in \Omega$  and  $\Omega \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$  (see Corollary 3.1) and the proof is complete.  $\blacksquare$

**Corollary 3.2** *Given  $A_0, A_g \in \mathbb{R}^{n \times n}$  such that  $A_0$  is Hurwitz, let the interval  $\mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$  as in Definition 3.1. This is the largest interval of  $\mathbb{R}$  containing the origin for which the matrix  $A_0 + \rho A_g$  is Hurwitz.*

*Proof.* Let  $\mathcal{L}_0$  denote the largest interval of  $\mathbb{R}$  containing the origin for which the matrix  $A_0 + \rho A_g$  is Hurwitz. It follows from Theorem 3.1 that  $\mathcal{L}_0 \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$ . Assume now that  $\rho \in \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$  and suppose that  $A(\rho)$  is not Hurwitz, i.e.,  $\text{Re}[\lambda_k A(\rho)] \geq 0$  for some  $k \in \mathcal{I}_n$ . Without loss of generality, assume  $\rho > 0$  (the case  $\rho < 0$  being identical). Since  $0 \in \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$  (see Corollary 3.1) it follows that  $[0, \rho] \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$ . Since  $\text{Re}[\lambda_i(A_0)] < 0$  for every  $i \in \mathcal{I}_n$  and the eigenvalues of  $A(\rho)$  change continuously with  $\rho$  (see [38], Appendix D), there exists  $\rho_2 \in (0, \rho] \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$  such that  $\text{Re}[\lambda_k A(\rho_2)] = 0$  for some  $k \in \mathcal{I}_n$ . Tracing the same steps as in the second part in the proof of Theorem 3.1 one can show that  $\det(I + \rho_2 \bar{A}_0^{-1} \bar{A}_g) = 0$  and hence  $\rho_2 \notin \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$ , a contradiction. Therefore,  $A(\rho)$  must be Hurwitz and  $\rho \in \mathcal{L}_0$ . It follows that  $\mathcal{N}(\bar{A}_0^{-1} \bar{A}_g) \subseteq \mathcal{L}_0$  and the proof is complete.  $\blacksquare$

### 3.3.2 Improved Stability Condition for Single LTIPD Systems

The application of the stability condition of Theorem 3.1 is limited owing to the large number of computations required to calculate the inverse of the  $n^2 \times n^2$  matrix  $\bar{A}_0$ , especially when the system is of high order. This limitation can be overcome by using the guardian map of Remark 2.1 which involves the determinant of the bialternate sum of a matrix with

itself. The resulting improved stability condition requires the calculation of the inverses of an  $n \times n$  and an  $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$  matrix. Using the map induced by the bialternate sum, one can easily obtain the following robust stability condition, which can also be used to synthesize the maximal continuous robust stability interval that includes the origin.

**Theorem 3.2** *Given an open interval  $\Omega$  in  $\mathbb{R}$ , and two matrices  $A_0, A_g \in \mathbb{R}^{n \times n}$ , the following two statements are equivalent:*

- (i)  $0 \in \Omega$ , and  $A(\rho) := A_0 + \rho A_g$  is Hurwitz for all  $\rho \in \Omega$
- (ii)  $A_0$  is Hurwitz and  $0 \in \Omega \subseteq \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$

*Proof.* Recall from the definition of the bialternate sum for the matrix  $A(\rho)$ , that

$$\begin{aligned}\tilde{A}(\rho) &:= 2A(\rho) \star I = (2A_0 \star I) + \rho(2A_g \star I) \\ &= \tilde{A}_0 + \rho\tilde{A}_g.\end{aligned}\tag{40}$$

(i)  $\Rightarrow$  (ii): If  $A(\rho)$  is Hurwitz for all  $\rho \in \Omega$  and  $0 \in \Omega$ , then  $A_0$  is Hurwitz. Then, by Corollary 2.1,  $\det \tilde{A}_0 \neq 0$ , and  $\tilde{A}_0^{-1}$  exists. Furthermore, since  $A(\rho)$  is Hurwitz for all  $\rho \in \Omega$ , and again using Corollary 2.1,  $\det \tilde{A}(\rho) \neq 0$  for all  $\rho \in \Omega$ . Therefore,

$$\begin{aligned}0 \neq \det \tilde{A}(\rho) &= \det(\tilde{A}_0 + \rho\tilde{A}_g) = \det[\tilde{A}_0(I + \rho\tilde{A}_0^{-1}\tilde{A}_g)] \\ &= \det \tilde{A}_0 \det(I + \rho\tilde{A}_0^{-1}\tilde{A}_g), \quad \rho \in \Omega.\end{aligned}$$

Hence,  $\det(I + \rho\tilde{A}_0^{-1}\tilde{A}_g) \neq 0$  for all  $\rho \in \Omega$ . Using the Schur Decomposition Lemma [38], there exists a unitary matrix  $U \in \mathbb{C}^{\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)}$  and an upper triangular matrix  $T \in \mathbb{C}^{\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)}$  such that  $\tilde{A}_0^{-1}\tilde{A}_g = UTU^*$ . The diagonal elements of the matrix  $T$  are the eigenvalues of matrix  $\tilde{A}_0^{-1}\tilde{A}_g$ . Thus, we have that  $\det(I + \rho T) \neq 0$  for all  $\rho \in \Omega$ , which, in turn, implies that

$$\rho \lambda_i(\tilde{A}_0^{-1}\tilde{A}_g) \neq -1, \quad \forall i \in \mathcal{I}_{\frac{1}{2}n(n-1)}, \quad \forall \rho \in \Omega.\tag{41}$$

If  $\tilde{A}_0^{-1}\tilde{A}_g$  has no real eigenvalues or the only real eigenvalues lie at the origin then  $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) = (-\infty, \infty)$  and, trivially,  $\Omega \subseteq \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$ . If  $\tilde{A}_0^{-1}\tilde{A}_g$  has some non-zero real eigenvalues, then

the largest continuous interval which includes  $\rho = 0$  such that  $\rho\lambda_i(\tilde{A}_0^{-1}\tilde{A}_g) \neq -1$ , for all  $i = 1, 2, \dots, \frac{1}{2}n(n-1)$  is given by  $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$ . It follows from (41) that  $\Omega \subseteq \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$ . Furthermore, since  $A(\rho) = A_0 + \rho A_g$  is Hurwitz for all  $\rho \in \Omega$ ,  $\det(A_0 + \rho A_g) \neq 0$ . This implies that  $\det(I + \rho A_0^{-1}A_g) \neq 0$  for all  $\rho \in \Omega$ , and hence the largest continuous interval which includes  $\rho = 0$  for which  $\rho\lambda_i(A_0^{-1}A_g) \neq -1$  and  $i = 1, 2, \dots, n$  is given by  $\mathcal{N}(A_0^{-1}A_g)$ . It follows that  $\Omega \subseteq \mathcal{N}(A_0^{-1}A_g)$  and finally that  $\Omega \subseteq \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$ .

(ii)  $\Rightarrow$  (i): The proof follows by contradiction. Assume  $0 \in \Omega \subseteq \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$  and suppose  $A(\rho)$  is not Hurwitz for all  $\rho \in \Omega$ . Then, there exists a  $\rho_1 \in \Omega$  such that  $\operatorname{Re}[\lambda_i(A(\rho_1))] \geq 0$  for some  $i \in \mathcal{I}_n$ . If  $\rho_1 = 0$ , the proof is complete since  $A_0$  is assumed Hurwitz. Consequently, and without loss of generality, let us assume that  $\rho_1 > 0$  (the case  $\rho_1 < 0$  being identical). Since  $\operatorname{Re}[\lambda_i(A_0)] < 0$  for all  $i \in \mathcal{I}_n$  and the eigenvalues of  $A(\rho)$  change continuously with  $\rho$  (see [38], Appendix D), there exists  $\rho_2 \in (0, \rho_1] \subseteq \Omega$  such that  $\operatorname{Re}[\lambda_k(A(\rho_2))] = 0$  for some  $k \in \mathcal{I}_n$ . There are two possibilities:

1.  $\lambda_k(A(\rho_2)) = 0$ . This implies that  $\det(A_0 + \rho_2 A_g) = 0$  with, in turn, implies that  $\det(I + \rho_2 A_0^{-1}A_g) = 0$ , which cannot be satisfied when  $\rho_2 \in \Omega$  since  $\Omega \subseteq \mathcal{N}(A_0^{-1}A_g)$ , hence we get a contradiction.
2.  $\lambda_k(A(\rho_2)) = j\omega$  and  $\omega \neq 0$ . Since  $A(\rho_2) \in \mathbb{R}^{n \times n}$ , there exists  $k' \in \mathcal{I}_n$  such that  $\lambda_{k'}(A(\rho_2)) = -j\omega$  and hence by Lemma 2.1, there exists  $m \in \mathcal{I}_{\frac{1}{2}n(n-1)}$  such that  $\lambda_m(\tilde{A}(\rho_2)) = \lambda_k(A(\rho_2)) + \lambda_{k'}(A(\rho_2)) = 0$ . Consequently, there exists  $m \in \mathcal{I}_{\frac{1}{2}n(n-1)}$  such that  $\lambda_m(\tilde{A}(\rho_2)) = 0$  with  $\rho_2 \in \Omega$  and  $\det \tilde{A}(\rho_2) = 0$ . However, since  $A_0$  is Hurwitz, by Corollary 2.1,  $\tilde{A}_0^{-1}$  exists and we can write:

$$\begin{aligned} 0 &= \det \tilde{A}(\rho_2) = \det(\tilde{A}_0 + \rho_2 \tilde{A}_g) \\ &= \det \tilde{A}_0 \det(I + \rho_2 \tilde{A}_0^{-1} \tilde{A}_g), \quad \rho_2 \in \Omega \end{aligned}$$

This implies that  $\rho_2 \lambda_i(\tilde{A}_0^{-1} \tilde{A}_g) = -1$  for some  $i \in \mathcal{I}_{\frac{1}{2}n(n-1)}$ . This condition cannot be satisfied when  $\rho_2 \in \Omega$  and  $\Omega \subseteq \mathcal{N}(\tilde{A}_0^{-1} \tilde{A}_g)$  and hence the proof is complete.  $\blacksquare$

**Corollary 3.3** *Given  $A_0, A_g \in \mathbb{R}^{n \times n}$  such that  $A_0$  is Hurwitz, then  $\mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$*

is the largest continuous interval of  $\mathbb{R}$  containing the origin for which the matrix  $A_0 + \rho A_g$  is Hurwitz.

*Proof.* Let  $\mathcal{L}_0$  denote the largest interval of  $\mathbb{R}$  containing the origin for which the matrix  $A_0 + \rho A_g$  is Hurwitz. It follows from Theorem 3.2 that  $\mathcal{L}_0 \subseteq \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$ . Assume now that  $\rho \in \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$  and suppose that  $A(\rho)$  is not Hurwitz, i.e.,  $\text{Re}[\lambda_k A(\rho)] \geq 0$  for some  $k \in \mathcal{I}_n$ . Without loss of generality, assume  $\rho > 0$  (the case  $\rho < 0$  being identical). Since  $0 \in \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$  (see Corollary 3.1) it follows that  $[0, \rho] \subseteq \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$ . Since  $\text{Re}[\lambda_i(A_0)] < 0$  for every  $i \in \mathcal{I}_n$  and the eigenvalues of  $A(\rho)$  change continuously with  $\rho$  (see [38], Appendix D), there exists  $\rho_2 \in (0, \rho] \subseteq \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$  such that  $\text{Re}[\lambda_k A(\rho_2)] = 0$  for some  $k \in \mathcal{I}_n$ . Tracing the same steps as in the second part in the proof of Theorem 3.2 one can show that either  $\det(I + \rho_2 A_0^{-1}A_g) = 0$  or  $\det(I + \rho_2 \tilde{A}_0^{-1}\tilde{A}_g) = 0$ . Hence  $\rho_2 \notin \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$ , a contradiction. Therefore,  $A(\rho)$  must be Hurwitz and  $\rho \in \mathcal{L}_0$ . It follows that  $\mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \subseteq \mathcal{L}_0$  and the proof is complete. ■

The following result follows immediately from Corollary 3.2 and Corollary 3.3.

**Corollary 3.4** *Given  $A_0, A_g \in \mathbb{R}^{n \times n}$  suppose that  $A_0$  is Hurwitz. Then,*

$$\mathcal{N}(\bar{A}_0^{-1}\bar{A}_g) = \mathcal{N}(A_0^{-1}A_g) \cap \mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \quad (42)$$

## 3.4 Complete Stability Domain of Single LTIPD Systems

### 3.4.1 Stability Condition using the Kronecker Sum

Theorems 3.1 and 3.2 give the maximal continuous stability interval in  $\mathbb{R}$  which includes the origin. These two theorems provide nonetheless only sufficient conditions for a single-parameter dependent matrix to be Hurwitz, because in many cases the maximal stability interval around the origin is not the complete stability domain. Additionally, the requirement that  $A_0$  is Hurwitz limits the applicability of Theorems 3.1 and 3.2. In this section,

our objective is to obtain the complete stability domain without requiring  $A_0$  to be Hurwitz.

**Theorem 3.3** *Let  $A_0, A_g \in \mathbb{R}^{n \times n}$  with  $\det(A_0 \oplus A_0) \neq 0$ . If there exists a stability domain  $\Omega \subseteq \mathbb{R}$  such that  $A_0 + \rho A_g$  is Hurwitz for all  $\rho \in \Omega$ , then this domain  $\Omega$  is an open interval or a union of disjointed open intervals of  $\mathbb{R}$  and the number of such intervals is finite. Furthermore, this number is no larger than  $n^2 + 1$ .*

*Proof.* Since the eigenvalues  $\lambda_j(A_0 + \rho A_g)$ ,  $j = 1, 2, \dots, n$  vary continuously with the parameter  $\rho$  (see [38] Appendix D), if  $A_0 + \rho_i A_g$  is Hurwitz for some  $\rho_i \in \Omega$ , there exists  $\delta > 0$  such that  $A_0 + \rho A_g$  is Hurwitz for all  $\rho \in (\rho_i - \delta, \rho_i + \delta)$ . Therefore, if  $\Omega$  exists, it must be an open interval or a disjointed union of open intervals. Let  $\Omega$  be expressed as<sup>1</sup>  $\Omega = \bigcup_{i=1}^m (\underline{\rho}_i, \bar{\rho}_i)$ , where  $\underline{\rho}_i < \bar{\rho}_i$  and  $m$  is the (perhaps infinite) number of the disjointed open intervals composing  $\Omega$ . Since  $\Omega$  is the entire stability region of  $\rho$ , it follows that for every  $\underline{\rho}_i \in \mathbb{R}$ ,  $i \in \mathcal{I}_m$ ,  $\text{Re}[\lambda_k(A_0 + \underline{\rho}_i A_g)] = 0$  for some  $k \in \mathcal{I}_n$ . Hence, by Lemma 2.1,  $\lambda_{k'}(\bar{A}_0 + \underline{\rho}_i \bar{A}_g) = 0$  for some  $k' \in \mathcal{I}_{n^2}$  and hence  $\det(\bar{A}_0 + \underline{\rho}_i \bar{A}_g) = 0$ . Since  $\det(A_0 \oplus A_0) = \det \bar{A}_0 \neq 0$ ,  $\bar{A}_0^{-1}$  exists. Thus,

$$\det(I + \underline{\rho}_i \bar{A}_0^{-1} \bar{A}_g) = 0 \quad \forall i \in \mathcal{I}_m \quad (\text{excluding } i = 1 \text{ if } \underline{\rho}_i = -\infty). \quad (43)$$

Since this equation has a finite number of solutions,  $m < \infty$ . By Definition 3.2 and Eq. (43), it follows that  $\underline{\rho}_i \in \mathcal{B}(\bar{A}_0^{-1} \bar{A}_g)$ ,  $i \in \mathcal{I}_m$ . Similarly, one can show that  $\bar{\rho}_i \in \mathcal{B}(\bar{A}_0^{-1} \bar{A}_g)$ ,  $i \in \mathcal{I}_m$ . Therefore,

$$m \leq \mathcal{B}^\#(\bar{A}_0^{-1} \bar{A}_g) - 1 \quad (44)$$

where  $\mathcal{B}^\#(\bar{A}_0^{-1} \bar{A}_g)$  stands for the cardinality of the set  $\mathcal{B}(\bar{A}_0^{-1} \bar{A}_g)$ . From the definition of the set  $\mathcal{B}(\bar{A}_0^{-1} \bar{A}_g)$  it is clear that  $\mathcal{B}^\#(\bar{A}_0^{-1} \bar{A}_g) \leq n^2 + 2$ . Using (44) it follows that  $m \leq n^2 + 1$ . ■

**Theorem 3.4** *Let  $A_0, A_g \in \mathbb{R}^{n \times n}$  with  $\det(A_0 \oplus A_0) \neq 0$ . Define  $\bar{A}_0 := A_0 \oplus A_0$  and  $\bar{A}_g := A_g \oplus A_g$  and let  $p = \mathcal{B}^\#(\bar{A}_0^{-1} \bar{A}_g) - 2$ . Suppose there exists a real number  $\rho_i \in (r_i, r_{i+1})$ ,*

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<sup>1</sup>With the possibility that  $\underline{\rho}_1 = -\infty$  and  $\bar{\rho}_m = +\infty$ .

where  $r_i, r_{i+1}$  are consecutive members of  $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$ ,  $i \in \mathcal{I}_p^0$  such that  $A_0 + \rho_i A_g$  is Hurwitz. Then  $A_0 + \rho A_g$  is Hurwitz for all  $\rho \in (r_i, r_{i+1})$ .

*Proof.* The map  $\nu_1: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$  given by

$$\nu_1(A) = \det(A \oplus A)$$

is a guardian map for the set  $\mathcal{A}$  of stable  $n \times n$  matrices (see page 303 of [7]). Let  $A(\rho) := A_0 + \rho A_g$ . According to the definition of  $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$ , if  $r_i, r_{i+1}$  are consecutive members of  $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$ , such that  $r_i, r_{i+1} \neq \pm\infty$ , then  $\nu_1(A(r_i)) = 0$  and  $\nu_1(A(r_{i+1})) = 0$ . Furthermore,  $\nu_1(A(\rho)) \neq 0$  for all  $r_i < \rho < r_{i+1}$ . Let now some  $\rho_i \in (r_i, r_{i+1})$  such that  $A_0 + \rho_i A_g$  is Hurwitz. Since  $\nu_1$  is a guardian map, it follows that  $A(\rho)$  is Hurwitz for all  $\rho \in (r_i, r_{i+1})$ . ■

**Theorem 3.5** Given  $A_0, A_g \in \mathbb{R}^{n \times n}$  with  $\det(A_0 \oplus A_0) \neq 0$ , let  $\bar{A}_0 := A_0 \oplus A_0$ ,  $\bar{A}_g := A_g \oplus A_g$  and let  $p = \mathcal{B}^\#(\bar{A}_0^{-1}\bar{A}_g) - 2$ . Define the index set

$$\begin{aligned} \mathcal{I} := \{ & i \in \mathcal{I}_p^0 : A_0 + \rho_i A_g \text{ is Hurwitz for some } \rho_i \in (r_i, r_{i+1}), \\ & r_i, r_{i+1} \text{ consecutive members of } \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g) \} \end{aligned} \quad (45)$$

and the open set

$$\Omega_\epsilon := \bigcup_{i \in \mathcal{I}} (r_i, r_{i+1}) \quad (46)$$

Then,  $A_0 + \rho A_g$  is Hurwitz if and only if  $\rho \in \Omega_\epsilon$ .

*Proof.* To prove sufficiency, choose  $\rho \in \Omega_\epsilon$  and let  $\rho \in (r_i, r_{i+1})$  for some  $i \in \mathcal{I}$ . From Theorem 3.4 and the fact that  $A_0 + \rho_i A_g$  is Hurwitz for  $\rho_i \in (r_i, r_{i+1})$ , it follows that  $A_0 + \rho A_g$  is Hurwitz. To prove necessity, assume that  $A_0 + \rho A_g$  is Hurwitz. It follows that  $\rho \notin \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$ . Therefore, there exists  $i \in \mathcal{I}_p^0$  such that  $r_i < \rho < r_{i+1}$ . Since  $A_0 + \rho A_g$  is Hurwitz, it follows that  $i \in \mathcal{I}$ . Hence,  $\rho \in (r_i, r_{i+1}) \subseteq \Omega_\epsilon$ . ■

**Remark 3.2** Theorem 3.5 can be used to find the *exact* stability domain  $\Omega_\epsilon$  for a parameter-dependent matrix  $A(\rho) = A_0 + \rho A_g$  where  $\rho \in \mathbb{R}$  and  $A_0, A_g \in \mathbb{R}^{n \times n}$ . The procedure involves four steps.

1. Calculate  $\bar{A}_0, \bar{A}_g$  and the eigenvalues of the matrix  $\bar{A}_0^{-1} \bar{A}_g$ .
2. Choose the real, distinct, non-zero eigenvalues of the matrix  $\bar{A}_0^{-1} \bar{A}_g$  and construct the set  $\mathcal{B}(\bar{A}_0^{-1} \bar{A}_g)$  according to Definition 3.2.
3. Check whether the matrix  $A_0 + \rho_i A_g$  is Hurwitz for *any*  $\rho_i \in (r_i, r_{i+1})$ ,  $i \in \mathcal{I}_p^0$ ,  $p = \mathcal{B}^\#(\bar{A}_0^{-1} \bar{A}_g) - 2$ , and construct the index set  $\mathcal{I}$ .
4. Let  $\Omega_\epsilon$  as in (46).

### 3.4.2 Stability Condition using the Bialternate Sum

The need to do intensive numerical calculations in order to calculate the inverse and the eigenvalues of the  $n^2 \times n^2$  matrix  $\bar{A}_0 = \det A_0 \oplus A_0$  limits the applicability of Theorem 3.5. This limitation can be overcome using a map induced by the bialternate sum of a matrix with itself (see (21) and Remark 2.1). In this case, it is only needed to calculate the inverse of a matrix of dimension  $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ .

**Theorem 3.6** *Let  $A_0, A_g \in \mathbb{R}^{n \times n}$  with  $\det(A_0 \oplus A_0) \neq 0$ . If there exists a stability domain  $\Omega \subseteq \mathbb{R}$  such that  $A_0 + \rho A_g$  is Hurwitz for all  $\rho \in \Omega$ , then this domain  $\Omega$  is an open interval or a union of disjointed open intervals of  $\mathbb{R}$ , and the number of such intervals is finite. Furthermore, this number is no greater than  $\frac{1}{2}(n^2 + n + 2)$ .*

*Proof.* The fact that the stability domain is an open interval or a union of disjointed open intervals follows from the proof of Theorem 3.3.  $\Omega$  can therefore be expressed as  $\Omega = \bigcup_{i=1}^m (\underline{\rho}_i, \bar{\rho}_i)$ , where  $\underline{\rho}_i < \bar{\rho}_i$  and  $m$  is the number of the disjointed open intervals composing  $\Omega$ . Since  $\Omega$  is the entire stability region of  $\rho$ , it follows that for every  $\underline{\rho}_i \in \mathbb{R}$ ,  $i \in \mathcal{I}_n$ ,  $\text{Re}[\lambda_k(A_0 + \underline{\rho}_i A_g)] = 0$  for some  $k \in \mathcal{I}_n$ . Following an argument similar to the one



in the proof of Theorem 3.2, one can show that  $\underline{\rho}_i \in \mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$  for all  $i \in \mathcal{I}_m$ . Similarly, one can show that for every  $\bar{\rho}_i \in \mathbb{R}, i \in \mathcal{I}_m$ ,  $\text{Re}[\lambda_k(A_0 + \bar{\rho}_i A_g)] = 0$  for some  $k \in \mathcal{I}_n$ . Then  $\bar{\rho}_i \in \mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$  for all  $i \in \mathcal{I}_m$ . Therefore,

$$m \leq (\mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g))^{\#} - 1 \quad (47)$$

From the definition of the sets  $\mathcal{B}(A_0^{-1}A_g)$  and  $\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$ , it is clear that  $\mathcal{B}^{\#}(A_0^{-1}A_g) \leq n+2$  and  $\mathcal{B}^{\#}(\tilde{A}_0^{-1}\tilde{A}_g) \leq \frac{1}{2}n(n-1) + 2$ . Using (47) and the fact that  $\{-\infty, +\infty\}$  belongs to both sets, it follows that  $m \leq \frac{1}{2}(n^2 + n + 2)$ .  $\blacksquare$

**Remark 3.3** Since  $\frac{1}{2}(n^2 + n + 2) \leq (n^2 + 1)$  for all  $n \geq 1$ , Theorem 3.6 gives a better estimate for the number of stability intervals than Theorem 3.3.

**Theorem 3.7** *Given  $A_0, A_g \in \mathbb{R}^{n \times n}$  with  $\det(A_0 \oplus A_0) \neq 0$  let  $\tilde{A}_0 := 2A_0 \star I$  and  $\tilde{A}_g := 2A_g \star I$ . Let  $p = (\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g) \cup \mathcal{B}(A_0^{-1}A_g))^{\#} - 2$ . Suppose there exists a real number  $\rho_i \in (r_i, r_{i+1})$ , where  $r_i, r_{i+1}$  are consecutive members of  $\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g) \cup \mathcal{B}(A_0^{-1}A_g)$ ,  $i \in \mathcal{I}_p^0$ , such that  $A_0 + \rho_i A_g$  is Hurwitz. Then  $A_0 + \rho A_g$  is Hurwitz for all  $\rho \in (r_i, r_{i+1})$ .*

*Proof.* The map  $\nu_2: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$  given by

$$\nu_2(A) = \det A \det \tilde{A}$$

is a guardian map for the set  $\mathcal{A}$  of stable  $n \times n$  matrices (see Remark 2.1). Since  $\det(A_0 \oplus A_0) \neq 0$ ,  $\det A_0 \neq 0$  and  $\det \tilde{A}_0 \neq 0$ . Let  $A(\rho) := A_0 + \rho A_g$ . Then,

$$\begin{aligned} \nu_2(A(\rho)) &= \det A(\rho) \det \tilde{A}(\rho) \\ &= \det(A_0 + \rho A_g) \det(\tilde{A}_0 + \rho \tilde{A}_g) \\ &= \det A_0 \det \tilde{A}_0 \det(I + \rho A_0^{-1}A_g) \det(I + \rho \tilde{A}_0^{-1}\tilde{A}_g). \end{aligned}$$

According to the definition of  $\mathcal{B}(A_0^{-1}A_g)$  and  $\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$ , if  $r_i, r_{i+1} \in \mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$  and  $r_i, r_{i+1} \neq \pm\infty$ ,  $\nu_2(A(r_i)) = 0$  and  $\nu_2(A(r_{i+1})) = 0$ . Furthermore,  $\nu_2(A(\rho)) \neq 0$  if  $r_i < \rho < r_{i+1}$ . Let now some  $\rho_i \in (r_i, r_{i+1})$  be such that  $A_0 + \rho_i A_g$  is Hurwitz. Since  $\nu_2$  is a guardian map, it follows that  $A(\rho)$  is Hurwitz for all  $\rho \in (r_i, r_{i+1})$ .  $\blacksquare$

**Theorem 3.8** Given  $A_0, A_g \in \mathbb{R}^{n \times n}$  with  $\det(A_0 \oplus A_0) \neq 0$  let  $\tilde{A}_0 := 2A_0 \star I$  and  $\tilde{A}_g := 2A_g \star I$ . Let the integer  $p = (\mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g))^{\#} - 2$  and define the index set

$$\begin{aligned} \mathcal{I} := \{ i \in \mathcal{I}_p^0 : A_0 + \rho_i A_g \text{ is Hurwitz for some } \rho_i \in (r_i, r_{i+1}), \text{ where} \\ r_i, r_{i+1} \text{ are consecutive members of } \mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g) \} \end{aligned} \quad (48)$$

and the open set

$$\Omega_\epsilon := \bigcup_{i \in \mathcal{I}} (r_i, r_{i+1}) \quad (49)$$

Then,  $A_0 + \rho A_g$  is Hurwitz if and only if  $\rho \in \Omega_\epsilon$ .

*Proof.* To prove sufficiency, choose  $\rho \in \Omega_\epsilon$  and let  $\rho \in (r_i, r_{i+1})$  for some  $i \in \mathcal{I}$ . From Theorem 3.7 and the fact that  $A_0 + \rho_i A_g$  is Hurwitz for  $\rho_i \in (r_i, r_{i+1})$ , it follows that  $A_0 + \rho A_g$  is Hurwitz. To prove necessity, assume that  $A_0 + \rho A_g$  is Hurwitz. It follows that  $\nu_2(A(\rho)) = \det A_0 \det \tilde{A}_0 \det(I + \rho A_0^{-1}A_g) \det(I + \rho \tilde{A}_0^{-1}\tilde{A}_g) \neq 0$ . This implies that  $\rho \notin \mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$ . Therefore there must exist  $i \in \mathcal{I}_p^0$  such that  $r_i < \rho < r_{i+1}$  with  $r_i, r_{i+1}$  being consecutive members of  $\mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$ . Since  $A_0 + \rho A_g$  is Hurwitz, it follows that  $i \in \mathcal{I}$  and  $\rho \in (r_i, r_{i+1}) \subseteq \Omega_\epsilon$ .  $\blacksquare$

### 3.5 Stability Condition for Multi LTIPD Systems

In this section, the robust stability condition for the following multi-parameter dependent LTI system will be studied

$$\dot{x} = (A_0 + \sum_{i=1}^k \rho_i A_{g,i})x. \quad (50)$$

Reference [63] gives a stability condition for a system of the form (50), however that condition is only sufficient. Saydy *et al.* [64] used a semi-guardian map<sup>2</sup> to investigate robust stability for the following two-parameter quadratically-dependent matrix over the domain

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<sup>2</sup>A map  $\nu$  from the set of  $n \times n$  real Hurwitz matrices  $\mathcal{A}$  onto  $\mathbb{R}$  is a semi-guardian map if it is continuous, not identically zero and  $A \in \partial\mathcal{A} \Rightarrow \nu(A) = 0$ .

$$(r_1, r_2) \in [0, 1] \times [0, 1]$$

$$A(\rho_1, \rho_2) = \sum_{i_1, i_2=0}^{i_1+i_2=m} \rho_1^{i_1} \rho_2^{i_2} A_{i_1, i_2}. \quad (51)$$

The stability test in [64] requires the parameter domain to be known a priori. Consequently, the test checks whether the matrix is Hurwitz for all values of the parameters in a *given* domain. In this section, we extend the results of Section 3.4.2 to synthesize the entire stability region.

**Lemma 3.1** *Given the vector  $(\rho_1, \rho_2, \dots, \rho_k)^T \in \mathbb{R}^k$ ,  $k \geq 2$ , there exists a real number  $r$  and  $k-1$  scalars  $\theta_i \in [0, \pi)$ ,  $i = 2, \dots, k$  such that*

$$(\rho_1, \rho_2, \dots, \rho_k)^T = r_k v_k(\theta) \quad (52)$$

where  $\theta := (\theta_2, \dots, \theta_k)^T \in [0, \pi)^{k-1}$  and

$$\begin{aligned} v_k(\theta) &:= (\sin \theta_2, \cos \theta_2 \sin \theta_3, \cos \theta_2 \cos \theta_3 \sin \theta_4, \dots, \cos \theta_2 \cos \theta_3 \cdots \cos \theta_{k-1} \sin \theta_k, \\ &\quad \cos \theta_2 \cos \theta_3 \cdots \cos \theta_k)^T \in \mathbb{R}^k \end{aligned} \quad (53)$$

*Proof.* The proof follows by induction.

1. Let  $k = 2$ . Define

$$r = r_2 = \pm \sqrt{\rho_1^2 + \rho_2^2} \quad \text{and} \quad \theta_2 = \sin^{-1} \left( \frac{\rho_1}{\sqrt{\rho_1^2 + \rho_2^2}} \right) \quad (54)$$

Let  $v_2(\theta) = (\sin \theta_2, \cos \theta_2)^T$ . Clearly,  $(\rho_1, \rho_2) = r_2 v_2(\theta)$  as required.

2. Suppose for  $k > 2$ , we have that

$$\begin{aligned} (\rho_2, \rho_3, \dots, \rho_{k+1})^T &= r_k \left( \sin \theta_2, \cos \theta_2 \sin \theta_3, \cos \theta_2 \cos \theta_3 \sin \theta_4, \dots, \cos \theta_2 \cdots \cos \theta_{k-1} \sin \theta_k, \right. \\ &\quad \left. \cos \theta_2 \cos \theta_3 \cdots \cos \theta_k \right)^T = r_k v_k(\theta) \end{aligned} \quad (55)$$

where  $\theta = (\theta_2, \theta_3, \dots, \theta_k) \in [0, \pi)^{k-1}$ .

3. For  $k+1$ , we have that:

$$\begin{aligned} (\rho_1, \rho_2, \dots, \rho_k, \rho_{k+1}) &= \left( \rho_1, (\rho_2, \dots, \rho_k, \rho_{k+1}) \right) \\ &= \left( \rho_1, r_k v_k(\theta) \right) \end{aligned} \quad (56)$$

Let  $r_{k+1} = \pm \sqrt{\rho_1^2 + r_k^2}$  and  $\beta_2 = \sin^{-1} \left( \frac{\rho_1}{\sqrt{\rho_1^2 + r_k^2}} \right)$ , then we have:

$$\begin{aligned} (\rho_1, \rho_2, \dots, \rho_k, \rho_{k+1}) &= r_{k+1} \left( \sin \beta_2, \cos \beta_2 v_k(\theta) \right) \\ &= r_{k+1} \left( \sin \beta_2, \cos \beta_2 \sin \theta_2, \cos \beta_2 \cos \theta_2 \cdots \cos \theta_{k-1} \sin \theta_k, \right. \\ &\quad \left. \cos \beta_2 \cos \theta_2 \cdots \cos \theta_k \right) \end{aligned} \quad (57)$$

Letting now  $\beta_{i+1} = \theta_i$ ,  $i = 2, 3, \dots, k$ , then

$$(\rho_1, \rho_2, \dots, \rho_k, \rho_{k+1}) = r_{k+1} v_{k+1}(\beta)$$

where,  $\beta := (\beta_2, \dots, \beta_k, \beta_{k+1})^T \in [0, \pi)^k$ , and

$$\begin{aligned} v_{k+1}(\beta) &:= (\sin \beta_2, \cos \beta_2 \sin \beta_3, \cos \beta_2 \cos \beta_3 \sin \beta_4 \dots, \cos \beta_2 \cos \beta_3 \cdots \cos \beta_k \sin \beta_{k+1}, \\ &\quad \cos \beta_2 \cos \beta_3 \cdots \cos \beta_{k+1})^T \in \mathbb{R}^{k+1} \end{aligned}$$

The proof is complete. ■

We now use the stability condition of Theorem 3.5, to obtain the following stability condition for the dynamic system in (50).

**Theorem 3.9** *Given  $A_0, A_{g,i} \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, k$  with  $\det(A_0 \oplus A_0) \neq 0$ , define  $\bar{A}_0 := A_0 \oplus A_0$  and let  $(\rho_1, \rho_2, \dots, \rho_k)^T = rv(\theta)$  as in Lemma 3.1. Let  $p = \mathcal{B}^\#(\bar{A}_0^{-1} \bar{A}_g(\theta)) - 2$ ,  $A_g(\theta) := \sum_{i=1}^k A_{g,i} v_i(\theta)$ ,  $\bar{A}_g(\theta) := A_g(\theta) \oplus A_g(\theta)$ , and define the following set:*

$$\Omega_\epsilon(\theta) = \bigcup_{i \in \mathcal{I}(\theta)} (r_i, r_{i+1}) \quad (58)$$

where the index set  $\mathcal{I}(\theta)$  is given by

$$\begin{aligned} \mathcal{I}(\theta) &= \{i \in \mathcal{I}_p^0 : A_0 + r'_i A_g(\theta) \text{ is Hurwitz for some } r'_i \in (r_i, r_{i+1}), \text{ where} \\ &\quad r_i, r_{i+1} \text{ are consecutive members of } \mathcal{B}(\bar{A}_0^{-1} \bar{A}_g(\theta))\}. \end{aligned}$$

Let

$$\Omega'_\epsilon := \bigcup_{\theta \in [0, \pi)^{k-1}} \left\{ y(\theta) \in \mathbb{R}^k : y(\theta) = rv(\theta), r \in \Omega_\epsilon(\theta) \right\} \quad (59)$$

Then  $A_0 + \sum_{i=1}^k \rho_i A_{g,i}$  is Hurwitz if and only if  $(\rho_1, \dots, \rho_k)^T \in \Omega'_\epsilon$ .

*Proof.* Applying Lemma 3.1,  $(\rho_1, \dots, \rho_k)^T \in \mathbb{R}^k$  can be expressed as  $(\rho_1, \dots, \rho_k)^T = r(v_1(\theta), \dots, v_k(\theta))^T$ . The system matrix in equation (50) can then be rewritten as:

$$A_0 + \sum_i^k \rho_i A_{g,i} = A_0 + r \sum_{i=1}^k A_{g,i} v_i(\theta) = A_0 + r A_g(\theta) \quad (60)$$

When the angle vector  $\theta \in [0, \pi)^{k-1}$  is given, the system matrix in (60) is a single-parameter matrix which depends on  $r \in \mathbb{R}$ . Applying Theorem 3.5, the complete stability domain for  $r$  in the direction  $\theta$  can be calculated as in (58). The set defined by (59) is the union of the exact stability domains for the parameter  $r$  for every  $\theta \in \mathbb{R}^{k-1}$ . Therefore  $\Omega'_\epsilon$  is the exact stability domain for  $(\rho_1, \rho_2, \dots, \rho_k)^T \in \mathbb{R}^k$ . ■

**Theorem 3.10** *Given  $A_0, A_{g,i} \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, k$  with  $\det(A_0 \oplus A_0) \neq 0$ , define  $\tilde{A}_0 := 2A_0 \star I$  and let  $(\rho_1, \rho_2, \dots, \rho_k)^T = rv(\theta)$  as in Lemma 3.1. Let  $p = (\mathcal{B}(A_0^{-1}A_g(\theta)) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g(\theta)))^\# - 2$ ,  $A_g(\theta) := \sum_{i=1}^k A_{g,i}v_i(\theta)$ ,  $\tilde{A}_g(\theta) := 2A_g(\theta) \star I_n$ , and define the following open set:*

$$\Omega_\epsilon(\theta) = \bigcup_{i \in \mathcal{I}(\theta)} (r_i, r_{i+1}) \quad (61)$$

where the index set  $\mathcal{I}(\theta)$  is given by

$$\begin{aligned} \mathcal{I}(\theta) = \{ & i \in \mathcal{I}_p^0 : A_0 + r'_i A_g(\theta) \text{ is Hurwitz for some } r'_i \in (r_i, r_{i+1}), \text{ where} \\ & r_i, r_{i+1} \text{ are consecutive members of } \mathcal{B}(A_0^{-1}A_g(\theta)) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g(\theta)) \}. \end{aligned}$$

Let

$$\Omega'_\epsilon := \bigcup_{\theta \in [0, \pi)^{k-1}} \left\{ y(\theta) \in \mathbb{R}^k : y(\theta) = rv(\theta), r \in \Omega_\epsilon(\theta) \right\} \quad (62)$$

Then  $A_0 + \sum_{i=1}^k \rho_i A_{g,i}$  is Hurwitz if and only if  $(\rho_1, \dots, \rho_k)^T \in \Omega'_\epsilon$ .

*Proof.* The proof is similar to the one of Theorem 3.9 and thus, it is omitted. ■

Theorems 3.9 and 3.10 give the complete stability domain for multi parameter-dependent matrices. Moreover, these two results do not require that the matrix  $A_0$  is Hurwitz. The drawback of the approach is that the calculation of  $\Omega'_\epsilon$  requires, in general, gridding of the space  $[0, \pi)^{k-1}$ .

### 3.6 Numerical Examples

In the following examples, the stability domain for the matrix  $A(\rho) = A_0 + \rho A_g$ , with  $A_0, A_g \in \mathbb{R}^{n \times n}$ ,  $\rho \in \mathbb{R}$ , will be calculated by means of Theorems 3.1 and 3.2.

**Example 3.1** Consider the system matrix  $A(\rho) = A_0 + \rho A_g$  with

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_g = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since the matrix  $A(\rho)$  is upper triangular, the eigenvalues of  $A(\rho)$  are always  $\{-1, -1\}$ .

Hence, the largest stability domain for this example is  $(-\infty, +\infty)$ .

From Theorem 3.1, we calculate

$$\bar{A}_0^{-1} \bar{A}_g = \begin{bmatrix} 0 & -0.5 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and  $\text{mspec}(\bar{A}_0^{-1} \bar{A}_g) = \{0, 0, 0, 0\}$ . The largest continuous interval of  $\rho$  which includes zero and guarantees stability for the matrix  $A(\rho)$  is  $(-\infty, +\infty)$ . This agrees with the eigenvalue analysis.

Using Theorem 3.2, we have  $\tilde{A}_0 = -2$ ,  $\tilde{A}_g = 0$ ,  $\tilde{A}_0^{-1} \tilde{A}_g = 0$  and

$$\begin{aligned} \mathcal{N}(\tilde{A}_0^{-1} \tilde{A}_g) &= \mathcal{N}(0) = (-\infty, +\infty) \\ \mathcal{N}(A_0^{-1} A_g) &= \mathcal{N}\left(\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\right) = (-\infty, +\infty). \end{aligned}$$

The stability domain is  $\mathcal{N}(\tilde{A}_0^{-1} \tilde{A}_g) \cap \mathcal{N}(A_0^{-1} A_g) = (-\infty, +\infty)$ , which coincides with the result from Theorem 3.1 and the direct eigenvalue analysis.

**Example 3.2** Consider the system matrix  $A(\rho) = A_0 + \rho A_g$ , where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since the eigenvalues of matrix  $A(\rho)$  are  $\lambda_{1,2}(A(\rho)) = \{-2 \pm \rho i\}$ , the largest stability interval of  $\rho$  is  $(-\infty, +\infty)$ . Using Theorem 3.1, one obtains

$$\bar{A}_0^{-1} \bar{A}_g = \begin{bmatrix} 0 & -0.25 & -0.25 & 0 \\ 0.25 & 0 & 0 & -0.25 \\ 0.25 & 0 & 0 & -0.25 \\ 0 & 0.25 & 0.25 & 0 \end{bmatrix},$$

and  $\text{mspec}(\bar{A}_0^{-1} \bar{A}_g) = \{0, 0, -0.5i, 0.5i\}$ . The largest continuous interval of  $\rho$  which includes 0 that guarantees stability for  $A(\rho)$  is  $(-\infty, +\infty)$ .

Applying Theorem 3.2, one obtains that  $\tilde{A}_0 = -4$ ,  $\tilde{A}_g = 0$ ,  $\tilde{A}_0^{-1} \tilde{A}_g = 0$  and

$$\begin{aligned} \mathcal{N}(\tilde{A}_0^{-1} \tilde{A}_g) &= \mathcal{N}(0) = (-\infty, +\infty) \\ \mathcal{N}(A_0^{-1} A_g) &= \mathcal{N}\left(\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\right) = (-\infty, +\infty). \end{aligned}$$

The stability domain is  $\mathcal{N}(\tilde{A}_0^{-1} \tilde{A}_g) \cap \mathcal{N}(A_0^{-1} A_g) = (-\infty, +\infty)$ , which coincides with the result from Theorem 3.1 and the direct eigenvalue analysis.

**Example 3.3** Consider the matrix  $A(\rho) = A_0 + \rho A_g$ , where

$$A_0 = \begin{bmatrix} -2 & 0 \\ -3 & -2 \end{bmatrix}, \quad A_g = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of the matrix  $A(\rho)$  are given from the solution of the equation  $\lambda^2 + 4\lambda + (4 + 3\rho) = 0$ . Therefore if  $4 + 3\rho > 0$ , the matrix  $A(\rho)$  is Hurwitz. Hence the largest stability interval is  $(-1.333, \infty)$ .

Using Theorem 3.1, one obtains

$$\bar{A}_0^{-1} \bar{A}_g = \begin{bmatrix} 0 & -0.25 & -0.25 & 0 \\ 0 & 0.1875 & 0.1875 & -0.25 \\ 0 & 0.1875 & 0.1875 & -0.25 \\ 0 & -0.28125 & -0.28125 & 0.375 \end{bmatrix},$$

and  $\text{mspec}(\bar{A}_0^{-1}\bar{A}_g) = \{0, 0, 0.75, 0\}$ . The largest continuous interval of  $\rho$  which includes zero and guarantees stability for  $A(\rho)$  is  $(-1.333, +\infty)$ .

Using Theorem 3.2, one obtains  $\tilde{A}_0 = -4$ ,  $\tilde{A}_g = 0$ ,  $\tilde{A}_0^{-1}\tilde{A}_g = 0$ , and

$$\begin{aligned}\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) &= \mathcal{N}(0) = (-\infty, +\infty) \\ \mathcal{N}(A_0^{-1}A_g) &= \mathcal{N}\left(\begin{bmatrix} 0 & -0.5 \\ 0 & 0.75 \end{bmatrix}\right) = (-1.333, +\infty).\end{aligned}$$

The stability domain is  $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \cap \mathcal{N}(A_0^{-1}A_g) = (-\infty, +\infty) \cap (-1.333, +\infty) = (-1.333, +\infty)$ , which coincides with the result of Theorem 3.1 and the direct eigenvalue analysis.

**Example 3.4** Consider the matrix  $A(\rho) = A_0 + \rho A_g$ , where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Direct eigenvalue analysis of  $A(\rho)$  shows that  $\lambda_1 = -2 - \rho$  and  $\lambda_2 = -1 - \rho$ .

Using Theorem 3.1, one obtains

$$\bar{A}_0^{-1}\bar{A}_g = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and  $\text{mspec}(\bar{A}_0^{-1}\bar{A}_g) = \{0.5, 0.6667, 0.6667, 1\}$ . The largest continuous interval of  $\rho$  which includes zero and guarantees stability for  $A(\rho)$  is  $(-1, \infty)$  according to Theorem 3.1, which agrees with the eigenvalue analysis.

Using Theorem 3.2, one obtains  $\tilde{A}_0 = -3$ ,  $\tilde{A}_g = -2$ ,  $\tilde{A}_0^{-1}\tilde{A}_g = 2/3$  and

$$\begin{aligned}\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) &= \mathcal{N}(2/3) = (-3/2, +\infty) \\ \mathcal{N}(A_0^{-1}A_g) &= \mathcal{N}\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\right) = (-1, +\infty)\end{aligned}$$

The stability domain is  $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \cap \mathcal{N}(A_0^{-1}A_g) = (-1, +\infty) \cap (-\frac{3}{2}, +\infty) = (-1, +\infty)$ , which coincides with the result by Theorem 3.1 and the eigenvalue analysis.



**Example 3.5** Consider the matrix  $A(\rho) = A_0 + \rho A_g$ , where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Direct eigenvalue analysis of  $A(\rho)$ , gives  $\lambda_1 = \lambda_2 = -2 + \rho$ .

Using Theorem 3.1,

$$\bar{A}_0^{-1} \bar{A}_g = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 \end{bmatrix},$$

and  $\text{mspec}(\bar{A}_0^{-1} \bar{A}_g) = \{-0.5, -0.5, -0.5, -0.5\}$ . The largest continuous interval of  $\rho$  which includes 0 that guarantees stability for  $A(\rho)$  is  $(-\infty, 2)$  which agrees with the result from the eigenvalue analysis.

Using Theorem 3.2, one obtains  $\tilde{A}_0 = -4$ ,  $\tilde{A}_g = 2$ ,  $\tilde{A}_0^{-1} \tilde{A}_g = -0.5$  and

$$\begin{aligned} \mathcal{N}(\tilde{A}_0^{-1} \tilde{A}_g) &= \mathcal{N}(-0.5) = (-\infty, +2) \\ \mathcal{N}(A_0^{-1} A_g) &= \mathcal{N}\left(\begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}\right) = (-\infty, +2) \end{aligned}$$

The stability domain is  $\mathcal{N}(\tilde{A}_0^{-1} \tilde{A}_g) \cap \mathcal{N}(A_0^{-1} A_g) = (-\infty, +2) \cap (-\infty, +2) = (-\infty, +2)$ , which coincides with the result by Theorem 3.1 and the direct eigenvalue analysis.

**Example 3.6** Consider the matrix  $A(\rho) = A_0 + \rho A_g$ , where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Direct eigenvalue analysis of  $A(\rho)$  gives  $\lambda_1 = -2 + \rho$ ,  $\lambda_2 = -1 - \rho$ .

Using Theorem 3.1,

$$\bar{A}_0^{-1} \bar{A}_g = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and  $\text{mspec}(\bar{A}_0^{-1}\bar{A}_g) = \{-0.5, 0, 0, 1\}$ . The largest continuous interval of  $\rho$  which includes 0 that guarantees stability for  $A(\rho)$  is  $(-1, 2)$  which agrees with the eigenvalue analysis.

Using Theorem 3.2, one obtains  $\tilde{A}_0 = -3$ ,  $\tilde{A}_g = 0$ ,  $\tilde{A}_0^{-1}\tilde{A}_g = 0$  and

$$\begin{aligned}\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) &= \mathcal{N}(0) = (-\infty, +\infty) \\ \mathcal{N}(A_0^{-1}A_g) &= \mathcal{N}\left(\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\right) = (-1, +2).\end{aligned}$$

The stability domain is  $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \cap \mathcal{N}(A_0^{-1}A_g) = (-\infty, +\infty) \cap (-1, +2) = (-1, +2)$ , which coincides with the result of Theorem 3.1 and the direct eigenvalue analysis.

In order to compare Theorems 3.1, 3.2 and Theorems 3.5 and Theorem 3.8 we consider the following three examples.

**Example 3.7** Consider the matrix  $A(\rho) = A_0 + \rho A_g$ , where

$$A_0 = \begin{bmatrix} 62.563 & -121.34 & -217.75 & -111.86 & 309.77 \\ -64.806 & 123.09 & 214.78 & 115.44 & -319.39 \\ -7.6195 & 19.044 & 25.231 & 21.651 & -52.037 \\ 4.3314 & 1.9045 & -9.3643 & -3.8729 & 1.8837 \\ -44.276 & 91.392 & 150.51 & 85.741 & -235.05 \end{bmatrix}, \quad A_g = \begin{bmatrix} -5.9399 & -21.242 & 23.809 & 11.251 & -6.9852 \\ -8.8534 & -35.439 & 24.579 & 22.030 & 0.98018 \\ -10.049 & -21.452 & 20.026 & 13.640 & -4.3113 \\ 0.77706 & -24.138 & 15.174 & 9.3705 & 1.5890 \\ 2.2073 & -14.157 & 13.148 & 3.8678 & -8.9941 \end{bmatrix}.$$

Notice that for this example, both  $A_0$  and  $A_g$  are Hurwitz, but  $A_0 + A_g$  is not Hurwitz.

It is clear that in this case the maximal stability interval  $\Omega_\epsilon$  is composed of at least two disjointed open intervals. Theorems 3.1 and 3.2 give the maximal continuous stability domain  $(-0.02306, 0.11802)$  which includes the origin. Theorems 3.5 and 3.8 on the other hand, give the whole stability domain which is equal to  $(-0.02306, 0.11802) \cup (4.30818, +\infty)$ .

**Example 3.8** Consider the matrix  $A(\rho) = A_0 + \rho A_g$ , where

$$A_0 = \begin{bmatrix} -10.64 & 3.395 & 8.841 & 4.558 & -10.25 \\ -11.28 & -0.1536 & 14.67 & 9.852 & -13.53 \\ 0.7320 & 3.811 & -0.6074 & 2.408 & -10.44 \\ -12.14 & 4.938 & 9.649 & 1.152 & -6.297 \\ -11.66 & 6.451 & 11.70 & 9.453 & -17.28 \end{bmatrix}, \quad A_g = \begin{bmatrix} -110.9 & -247.0 & 162.4 & -57.61 & 194.2 \\ 241.82 & 731.3 & -446.6 & 87.68 & -511.8 \\ 366.8 & 987.5 & -617.4 & 181.9 & -777.1 \\ 385.3 & 1118.5 & -666.7 & 137.4 & -809.4 \\ 100.8 & 237.1 & -142.4 & 57.89 & -234.3 \end{bmatrix} \quad (63)$$

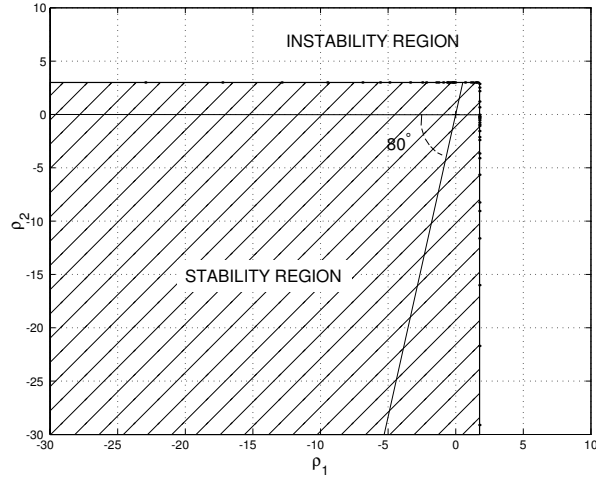
Theorems 3.1 and 3.2 give the maximal stability interval around the origin, which is  $(-0.04632, 0.00241)$ . Theorems 3.5 and 3.8 give the exact stability domain, which is  $(-0.04632, 0.00241) \cup (4.2279, +\infty)$ .

### 3.6.1 Multi-Parameter Case

**Example 3.9** This example is taken from [63]. Consider the matrix  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ , where

$$A(\rho) = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix} + \rho_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \rho_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The exact robust stability region for this problem is  $(-\infty, 1.75) \times (-\infty, 3)$  (see [63]). Theorem 3.9 or 3.10, give this exact stability domain as seen in Fig. 3. The same figure shows



**Figure 3:** Robust Stability Domain for Example 3.9

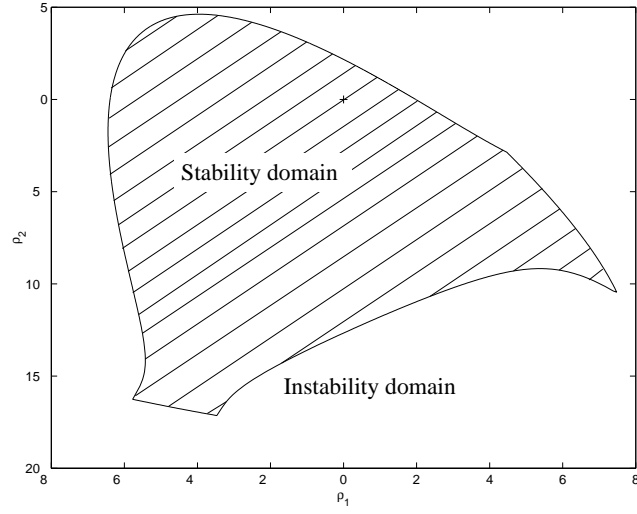
the exact stability region along a particular direction ( $\theta = 80^\circ$ ) which, for this case is

$$(\rho_1 \ \rho_2)^T \in \{\rho = r v(\theta) : r \in (-\infty, 3.0463), v(\theta) = (\cos 80^\circ \ \sin 80^\circ)^T\}. \quad (64)$$

**Example 3.10** Consider the matrix  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ , where

$$A_0 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.1509 & 0.8600 & 0.4966 \\ 0.6979 & 0.8537 & 0.8998 \\ 0.3784 & 0.5936 & 0.8216 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.6449 & 0.3420 & 0.5341 \\ 0.8180 & 0.2897 & 0.7271 \\ 0.6602 & 0.3412 & 0.3093 \end{bmatrix}$$

Theorems 3.9 and 3.10 give the same stability domain for the matrix  $A(\rho)$ , shown in Fig. 4.

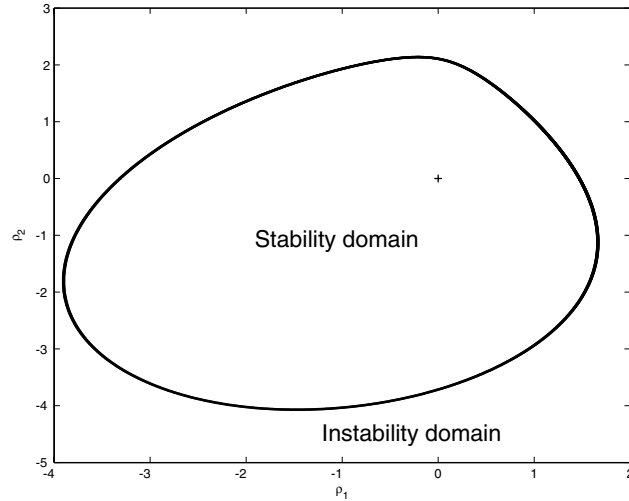


**Figure 4:** Robust Stability Domain for Example 3.10

**Example 3.11** Consider the matrix  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ , where

$$A_0 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.9802 & -0.003377 & -0.3599 \\ 0.5777 & -0.5720 & 0.9202 \\ -0.1227 & 0.2870 & 0.4533 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.2641 & -0.1802 & -0.8623 \\ 0.7337 & 1.300 & 1.018 \\ -0.6962 & 0.5500 & 0.3864 \end{bmatrix}$$

Theorem 3.9 and 3.10 give the same stability domain for the matrix  $A(\rho)$ , shown in Fig. 5.

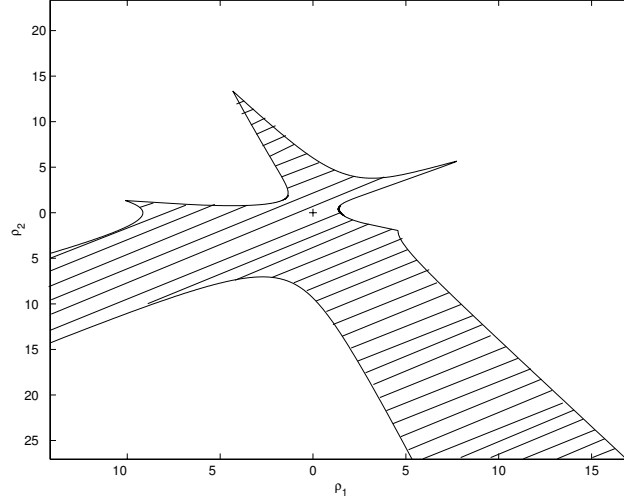


**Figure 5:** Robust Stability Domain for Example 3.11

**Example 3.12** Consider the matrix  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ , where

$$A_0 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.9160 & -0.8119 & -0.2168 \\ -0.6863 & -0.1001 & -0.4944 \\ -0.1673 & 0.7383 & -0.2912 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.215 & 1.664 & -2.209 \\ 0.7542 & -0.1501 & 0.2109 \\ 2.199 & 0.6493 & -0.2214 \end{bmatrix}$$

Theorem 3.9 and 3.10 give the same stability domain for matrix  $A(\rho)$ , shown in Fig. 6.



**Figure 6:** Robust Stability Domain for Example 3.12

**Example 3.13** Consider the matrix  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ , where

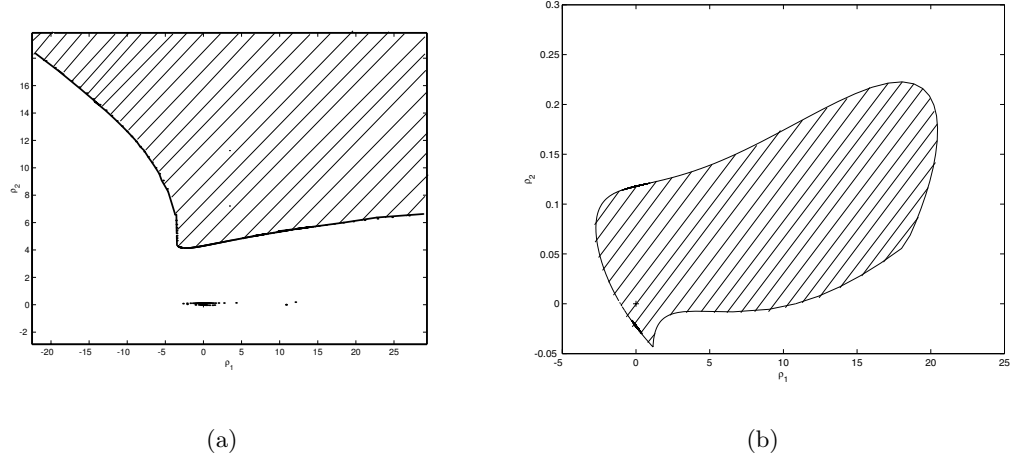
$$A_0 = \begin{pmatrix} 62.56 & -121.3 & -217.7 & -111.9 & 309.7e+002 \\ -64.81 & 123.1 & 214.78 & 115.4 & -319.4 \\ -7.619 & 19.04 & 25.23 & 21.651 & -52.04 \\ 4.331 & 1.904 & -9.364 & -3.873 & 1.884 \\ -44.28 & 91.39 & 150.5 & 85.74 & -235.0 \end{pmatrix} \quad (65)$$

$$A_1 = \begin{pmatrix} -0.1340 & 0.1139 & 0.2959 & 0.03392 & 0.2288 \\ 0.1747 & -0.2621 & 0.1509 & 0.2436 & 0.2165 \\ 0.1528 & 0.2313 & -0.06069 & 0.2725 & 0.1955 \\ 0.02228 & 0.09418 & 0.2484 & -0.2981 & 0.2262 \\ 0.05797 & 0.1914 & 0.2753 & 0.03664 & -0.1461 \end{pmatrix} \quad (66)$$

$$A_2 = \begin{pmatrix} -5.940 & -21.24 & 23.81 & 11.25 & -6.985 \\ -8.853 & -35.44 & 24.58 & 22.03 & 0.9802 \\ -10.05 & -21.45 & 20.03 & 13.64 & -4.311 \\ 0.7771 & -24.14 & 15.17 & 9.370 & 1.589 \\ 2.207 & -14.16 & 13.15 & 3.868 & -8.994 \end{pmatrix} \quad (67)$$

Both Theorems 3.9 and 3.10 give the same stability domain for the matrix  $A(\rho)$  shown

Fig. 7. In this case, the two-dimensional parameter stability space is composed of two disconnected sets. The area close the origin is zoomed in and is depicted in Fig. 7(b).



**Figure 7:** Robust Stability Domain for Example 3.13

### 3.7 Conclusions

In this chapter we have addressed the problem of stability for Linear Time Invariant Parameter Dependent (LTIPD) systems. We have extended previous results in the literature and have derived conditions that can be used to compute the exact stability region in the parameter space. We have made three specific contributions. The whole stability domain of the parameter  $\rho$  for the matrix  $A_0 + \rho A_g$  to be Hurwitz may be composed of several disconnected intervals in  $\mathbb{R}$ . The existing results require  $A_0$  to be Hurwitz and give only the maximal interval which includes  $\rho = 0$ . Our results do not require  $A_0$  to be Hurwitz and give the exact stability domain, which may be the union of several disconnected intervals in  $\mathbb{R}$ . For the LTIPD system (35) of dimension  $n$ , the current results use the guardian map induced by the Kronecker sum to calculate the stability domain. This requires the calculation of the eigenvalues of the inverse of an  $n^2 \times n^2$  matrix. When the system dimension  $n$  is large, this method is computationally intensive. The second contribution is the improvement of the current algorithm using the guardian map induced by the bialternate sum. With the new algorithm, and for the same dimension LTIPD system, one only needs to calculate the

eigenvalues of the inverse of a  $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$  matrix. The new algorithm has the benefit of requiring fewer computations compared to the existing one which uses the Kronecker sum. The third contribution is the generalization of this algorithm to the case when the parameter  $\rho$  is a vector, i.e., to the multi-parameter dependent system (36). However, in the multi-parameter case, our approach requires gridding which may limit the applicability of the results to low-parameter dimensions. Examples have been presented to demonstrate the applicability of the derived results.

## CHAPTER IV

### SINGLE-PARAMETER DEPENDENT LYAPUNOV FUNCTIONS FOR STABILITY ANALYSIS OF LTI SINGLE-PARAMETER DEPENDENT SYSTEMS

This chapter studies the stability problem of LTI Parameter-Dependent (LTIPD) systems with parameter-dependent Lyapunov matrices, and gives necessary and sufficient conditions for stability. For LTIPD systems of the form  $\dot{x} = A(\rho)x$ ,  $\rho \in \Omega$ , stability can be established via the use of constant Lyapunov functions, say, of the form  $V(x) = x^T P x$ . When the parameter  $\rho$  varies in the set  $\Omega$  or its value is not known a priori, a common (for all  $\rho$ ) Lyapunov function can be used to check Hurwitz stability of the family of matrices  $\{A(\rho), \rho \in \Omega\}$ . The resulting notion of stability (quadratic stability) is nonetheless conservative, since the same Lyapunov matrix  $P$  is used for the whole parameter space. The conservativeness of quadratic stability is more pronounced for the case of LTIPD systems where the parameter  $\rho$  does not vary with time. In order to achieve necessary and sufficient results one then needs to resort to the use of parameter-dependent Lyapunov functions of the form  $V(x, \rho) = x^T P(\rho)x$ .

Since the explicit dependence of the Lyapunov matrix  $P(\rho)$  on the parameter  $\rho$  is not known a priori, one typically postulates a convenient functional parameter dependence for  $P(\rho)$ , and then one proceeds to derive the stability conditions. This approach leads to conditions which are sufficient but not necessary [37, 12, 35, 40, 44, 56, 2, 73]. In order to obtain nonconservative (i.e., necessary and sufficient) conditions it is imperative to know the “correct” parameter dependence for the Lyapunov function. By “correct” we mean a Lyapunov function which depends on the parameter in such a way that for those values of the parameter for which the system is stable the stability conditions are satisfied, while for



the values of the parameter for which the system is not stable, the stability conditions fail.

This chapter shows that for LTI systems depending on a single, constant parameter in an affine manner, nonconservative stability tests can be derived by restricting the search over Lyapunov matrices which depend polynomially on the parameter. Therefore, a polynomial-type Lyapunov matrix (of known degree) is suggested, which can be used to derive necessary and sufficient stability conditions for single-parameter LTIPD systems.

The contributions of this chapter are summarized as follows: First, a polynomial-type Lyapunov function of bounded, computable degree is proposed which can be used to derive sufficient *and necessary* stability conditions for single-parameter LTIPD systems. These stability conditions are given in terms of two simultaneous matrix inequalities. The conditions take explicitly into consideration the rank deficiency of the system matrix multiplying the parameter in order to reduce the computational complexity of the proposed algorithm. In reference [88], Zelentsovsky, through the power transformation [19], developed sufficient conditions for the existence of the homogeneous polynomial Lyapunov functions of a given degree for a linear system with box-bounded uncertainty. The main contribution of this chapter is therefore the knowledge of the structure of the Lyapunov matrix that leads to non-conservative (i.e., exact) stability results for single-parameter LTIPD systems. This class of Lyapunov functions is simplified by lowering the polynomial degree of the Lyapunov function. This is achieved by taking into account the symmetry of the Lyapunov matrices. If the matrix multiplying the parameter is not full rank, the polynomial order can be reduced even further. Second, the inequalities for checking the stability of an LTIPD system over a compact interval are expressed into computable, non-conservative linear matrix inequalities (LMIs). It should be noted here that although the stability of LTIPD systems can also be checked using the guardian map techniques of [65, 24] (for similar results see also [94]) and Chapter 3 nonetheless, it is expected that the Lyapunov-based stability conditions of this chapter will be also amenable to synthesis. Such an extension to the synthesis problem is not directly evident from the use of guardian maps.

## 4.1 Linear Time Invariant Parameter Dependent Systems

The stability criteria for linear time-varying systems can be also applied to them. For the linear time-varying system, which is the solution of the homogenous differential equation

$$x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + a_2(t)x^{(n-2)}(t) + \dots + a_n(t)x(t) = 0$$

reference [75] gave a necessary and sufficient condition for the asymptotic stability. However, it involves the tests for sign definiteness of a certain matrix and a scalar function and leads to an existence test for functions satisfying differential inequalities. Under a certain restriction, a tractable sufficient criteria can be deducted from the necessary and sufficient stability condition.

It is desired to find a computable, non-conservative condition for checking the asymptotic stability of single-parameter dependent LTI system of the form

$$\dot{x} = A(\rho)x, \quad A(\rho) = A_0 + \rho A_1, \quad \rho \in \Omega \quad (68)$$

where  $A_0, A_1 \in \mathbb{R}^{n \times n}$  and  $\Omega \subset \mathbb{R}$ . At this point there is no a priori assumption on the set  $\Omega$  (i.e., connected, bounded, compact, etc.). The parameter  $\rho$  is assumed to be constant<sup>1</sup> and it is chosen from the set  $\Omega$ . It is well known that asymptotic stability of the system (68) is equivalent to the existence of a matrix  $P(\rho) \in \mathbb{R}^{n \times n}$  such that

$$P(\rho) > 0, \quad A^T(\rho)P(\rho) + P(\rho)A(\rho) < 0, \quad \rho \in \Omega. \quad (69)$$

The above inequality is the called Lyapunov inequality, and the existence of the positive definite matrix  $P(\rho)$  to the inequality (69) is equivalent to the existence of the positive definite matrix  $P(\rho)$  to the following *dual Lyapunov inequality*

$$P(\rho) > 0, \quad A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0, \quad \rho \in \Omega. \quad (70)$$

For convenience, (70), both the dual Lyapunov inequality and the dual Lyapunov equation will be used in the subsequent chapters. Thus, checking the stability of (68) is equivalent to

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<sup>1</sup>The results also hold when  $\rho$  varies very slowly so that a “quasi-static” point of view is valid.

finding a Lyapunov function  $P(\rho)$  satisfying the two matrix inequalities (70). By the same token, if for some  $\rho' \in \Omega$  the matrix  $A(\rho')$  is not Hurwitz, then there exists no positive-definite matrix that satisfies the second inequality in (70).

Clearly, for single-parameter LTIPD systems, as the one in equation (68), stability can be ensured if there exists a constant Lyapunov function  $P(\rho) = P$  for all  $\rho \in \Omega$ , such that the two inequalities (70) are satisfied. The so-called quadratic stability ensures robust stability against any (arbitrarily fast) variations of the parameter  $\rho$ . In case the parameters do not vary with time (such is the case with LTIDP systems), quadratic stability can be very conservative. To reduce this conservatism against slowly-varying or constant parameters, several parameter-dependent Lyapunov functions have been proposed in the literature [37, 12, 35, 40, 44, 56, 2, 73]. Such conditions, however, provide only sufficiency results which, in fact, may be far from necessary. On the other hand, for the multi- and single-parameter dependent LTI systems, references [22] and [55] provide a class of Lyapunov functions that can be used to derive necessary and sufficient conditions for system (68), assuming that the matrix  $A_1$  in (68) has rank one. The Lyapunov function proposed in [22] solves an augmented system and depends multiaffinely on the parameter vector. On the same token, in [55] it is shown that for a single parameter and for  $\text{rank } A_1 = 1$ , a Lyapunov function which is linear in the parameter can be used to characterize stability of the system (68).

More recently, [14, 13] proposed parameter-dependent Lyapunov functions of polynomial type in the parameter (of sufficiently high degree) which can be used to assess the robust stability of multi-linear systems over a compact set without conservatism. Recall that if  $Q(\rho) \in \mathbb{R}^{n \times n}$  is any positive-definite matrix for all  $\rho \in \Omega \subset \mathbb{R}$ , then the stability of (68) can be established by finding a positive-definite solution to the following Lyapunov equation

$$A(\rho)P(\rho) + P(\rho)A^T(\rho) + Q(\rho) = 0. \quad (71)$$

The solution  $P(\rho)$  to this equation is given explicitly as [98]

$$P(\rho) = \int_0^\infty e^{A(\rho)t} Q(\rho) e^{A^T(\rho)t} dt. \quad (72)$$

When  $Q(\rho)$  is analytic in  $\rho$ ,  $P(\rho)$  in Eq. (72) is also analytic in  $\rho$  and thus it can be

expressed in terms of power series in  $\rho$  as

$$P(\rho) = P_0 + \rho P_1 + \rho^2 P_2 + \dots + \rho^i P_i + \dots = \sum_{i=0}^{\infty} \rho^i P_i. \quad (73)$$

Starting from this simple observation, and using the uniform convergence of the integral in (72) at  $t = +\infty$  with respect to  $\rho$  when  $\Omega$  is compact, Bliman recently showed in [13] that the previous power series can be truncated and thus, a polynomial type Lyapunov matrix of the form

$$P(\rho) = P_0 + \rho P_1 + \rho^2 P_2 + \dots + \rho^m P_m = \sum_{i=0}^m \rho^i P_i \quad (74)$$

of sufficiently high degree  $m$  in  $\rho$  solves the Lyapunov inequality  $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$ , and thus it can be used to provide necessary and sufficient conditions for the robust stability of (68) over the set  $\Omega$ . In [13] however no a priori bound on the degree of the truncated polynomial is given. The main contribution of this chapter is to give an explicit bound for the polynomial dependence  $m$  of  $P(\rho)$  in  $\rho$  and to provide a computable algorithm for checking the associated linear matrix inequalities (70). In particular, it is shown that the existence of a Lyapunov matrix of the form (74) with  $m \leq \min\{\frac{1}{2}(2nr - r^2 + r), \frac{1}{2}n(n+1) - 1\}$  is necessary and sufficient for the stability of (68) for each  $\rho \in \Omega$ , where  $r$  denotes the rank of  $A_1$ . In other words, for every  $\rho \in \Omega$  if  $P(\rho)$  in (74) satisfies (70), then the matrix  $A(\rho)$  is Hurwitz. Most importantly, if for some  $\rho \in \Omega$  the matrix  $A(\rho)$  is not Hurwitz, then the matrix  $P(\rho)$  is either non-positive definite, or the second inequality in (70) does not hold. Finally, we show how the two matrix inequalities (70) involved in checking the stability of  $A(\rho)$  can be cast into computable LMIs without conservatism in case  $\Omega$  is a compact interval.

## 4.2 *Single-Parameter Dependent Lyapunov Functions*

### 4.2.1 Preliminaries

Given a polynomial  $p(\rho)$ , let  $\deg(p(\rho))$  denote the highest degree of  $p(\rho)$ .

**Lemma 4.1** *Let matrices  $A, B \in \mathbb{R}^{n \times n}$  with  $\text{rank } B = r$  and let  $\rho \in \mathbb{R}$ . Then  $\deg(\det(A + \rho B)) \leq r$ .*

*Proof.* Let the matrix  $B$  be decomposed as  $B = LR$  where  $L \in \mathbb{R}^{n \times r}$  a full column rank matrix and  $R \in \mathbb{R}^{r \times n}$  a full row rank matrix, respectively. Then, using the determinant formula

$$\det \begin{bmatrix} U & V \\ W & X \end{bmatrix} = \det(X) \det(U - VX^{-1}W),$$

we have

$$\det(A + \rho B) = \det(A + \rho LR) = \det(F) \quad \text{where} \quad F := \begin{bmatrix} A & \rho L \\ -R & I_r \end{bmatrix}. \quad (75)$$

Recall now that the determinant of a matrix  $F \in \mathbb{R}^{(n+r) \times (n+r)}$  can be computed from the formula [69]

$$\det(F) = \sum_{k \in \mathbf{K}} \text{sign}(k) \prod_{i=1}^{n+r} F_{i, k_i}$$

where  $k := (k_1, k_2, \dots, k_{n+r})$ ,  $\mathbf{K}$  is the set of permutations of  $\{1, \dots, n+r\}$ , and  $\text{sign}(k)$  is the signature of the permutation  $k$  taking the value either  $+1$  or  $-1$ . The determinant of  $F$  is thus a sum of  $(n+r)!$  terms, each term being the product of  $n+r$  elements. Moreover, each of these elements is chosen from a different row and column of the matrix  $F$ . Therefore, out of the  $n+r$  elements in each term, at most  $r$  elements can depend on  $\rho$  (in an affine manner), and thus we conclude that  $\det(F)$  is a polynomial in  $\rho$  of degree at most  $r$ . ■

The following lemma will play a major role in the results of this chapter. It states that the adjoint of the parameter-dependent matrix  $A + \rho B$  is a matrix polynomial in  $\rho$  of a certain maximal degree which depends on the rank of the matrix  $B$ . Recall that given an invertible matrix  $A \in \mathbb{R}^{n \times n}$ , its inverse can be calculated from  $A^{-1} = \text{Adj } A / \det(A)$  where  $\text{Adj } A$  is the adjoint of  $A$ .

**Lemma 4.2** *Given matrices  $A, B \in \mathbb{R}^{n \times n}$  with  $\text{rank } B = r$  and  $\rho \in \mathbb{R}$ , the adjoint of the*

matrix  $A + \rho B$  is a matrix polynomial in  $\rho$  of degree at most  $\min\{r, n-1\}$ , i.e.,

$$\text{Adj}(A + \rho B) = \sum_{i=0}^{\min\{r, n-1\}} \rho^i N_i. \quad (76)$$

*Proof.* From the definition of the adjoint matrix [38] it follows that

$$[\text{Adj}(A + \rho B)]_{ij} = (-1)^{i+j} \det(A + \rho B)_{[ji]} \quad (77)$$

where  $(\cdot)_{[ji]}$  is the  $(n-1) \times (n-1)$  submatrix of  $(\cdot)$  in which the  $j$ -th row and the  $i$ -th column are eliminated and  $[\cdot]_{ij}$  is the  $ij$ -th element of the matrix  $[\cdot]$ . Since  $\text{rank}(B_{[ji]}) \leq \text{rank } B = r$  it follows that

$$\text{rank}(B_{[ji]}) \leq \min\{r, n-1\}$$

From Lemma 4.1, and since  $(A + \rho B)_{[ji]} \in \mathbb{R}^{(n-1) \times (n-1)}$ , it follows that

$$\deg(\det(A + \rho B)_{[ji]}) = \deg(\det(A_{[ji]} + \rho B_{[ji]})) \leq \text{rank } B_{[ji]} \leq \min\{r, n-1\}, \quad (78)$$

where  $1 \leq i, j \leq n$ . From (77) and (78) it follows that  $\text{Adj}(A + \rho B)$  is a matrix-valued polynomial of degree  $\min\{r, n-1\}$  and hence, there exist matrices  $\{N_i\}_{i=0,1,\dots,\min\{r,n-1\}}$  such that (76) holds.  $\blacksquare$

Furthermore,  $N_i$ ,  $i = 1, 2, \dots, \min\{r, n-1\}$  in (76) can be calculated explicitly from the matrices  $A$  and  $B$  according to Corollary 2.2.

**Example 4.1** Consider the matrices

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 3 & -3 & 7 \\ -3 & -2 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 3 & -8 \\ -3 & 2 & 5 \end{bmatrix}. \quad (79)$$

Since  $\text{rank } B = 3 = n$ , the matrix  $B$  is full rank. Moreover, a straightforward calculation shows that

$$\text{Adj}(A + \rho B) = \begin{bmatrix} -13 - 18\rho + 31\rho^2 & \rho^2 - 68\rho - 17 & 6 + 36\rho - 48\rho^2 \\ 14\rho^2 - 30\rho - 48 & 29\rho^2 + 28\rho + 3 & 24\rho^2 + 23\rho - 29 \\ 13\rho^2 + 2\rho - 15 & -11\rho^2 - 20\rho - 5 & -3\rho^2 - 12\rho - 15 \end{bmatrix}.$$

In this case  $\text{Adj}(A + \rho B)$  has degree  $2 = n-1$ .

**Example 4.2** Consider the matrices

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 3 & -3 & 7 \\ -3 & -2 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 6 & 16 \\ 1 & 3 & 8 \end{bmatrix}. \quad (80)$$

Here  $\text{rank } B = 1 = r < n$  and thus  $B$  is rank deficient. The adjoint of  $A + \rho B$  is given by

$$\text{Adj}(A + \rho B) = \begin{bmatrix} -13 + 41\rho & -82\rho - 17 & 6 + 123\rho \\ -83\rho - 48 & 54\rho + 3 & -25\rho - 29 \\ 26\rho - 15 & -10\rho - 5 & -6\rho - 15 \end{bmatrix}.$$

In this case  $\text{Adj}(A + \rho B)$  has degree  $1 = r$ .

#### 4.2.2 A Class of Parameter-Dependent Lyapunov Functions for Single-Parameter LTIPD Systems

In this section we propose a class of parameter-dependent Lyapunov functions which can be used to test the stability of (68) in a non-conservative manner. Before doing that, some mathematical preliminaries are in order. In the sequel  $\text{mspec}(A)$  denotes the multispectrum of the matrix  $A$ , i.e., the set consisting of all the eigenvalues of  $A$ , including multiplicity.

With Lemma 2.1 and 2.2 in Chapter 2, we are now ready to state the main result of the paper. In the following,  $\Omega$  denotes any subset of  $\mathbb{R}$ .

**Theorem 4.1** *Given matrices  $A_0, A_1 \in \mathbb{R}^{n \times n}$  with  $\text{rank}(A_1) = r < n$ , the following two statements are equivalent.*

(i)  $A_0 + \rho A_1$  is Hurwitz for all  $\rho \in \Omega$ .

(ii) There exists a sequence of matrices  $P_i, i = 0, 1, \dots, 2nr - r^2$ , such that

$$(A_0 + \rho A_1)P(\rho) + P(\rho)(A_0 + \rho A_1)^T < 0, \quad \forall \rho \in \Omega \quad (81)$$

$$P(\rho) = \sigma(\rho) \sum_{i=0}^{2nr-r^2} \rho^i P_i > 0, \quad \forall \rho \in \Omega \quad (82)$$

where  $\sigma(\rho) = -\text{sign}(\det(\bar{A}_0 + \rho \bar{A}_1))$ .

Moreover, if  $\text{rank}(A_1) = n$ , the polynomial degree of  $P(\rho)$  in (82) will be  $n^2 - 1$ .

To prove Theorem 4.1 we will use the following two lemmas.

**Lemma 4.3** ([8]) *For any two matrices*

$$\text{rank}(A \otimes B) = \text{rank}(A) \text{rank}(B) \quad (83)$$

**Lemma 4.4** *Given  $M \in \mathbb{R}^{n \times n}$  with  $\text{rank}(M) = r$ , then  $\text{rank}(\bar{M}) \leq r(2n - r)$ .*

*Proof.* Let  $V \in \mathbb{R}^{n \times (n-r)}$  such that the columns of  $V$  form a basis of the null space  $\mathcal{N}(M)$  of  $M$  and thus,  $MV = 0$ . Notice now that  $(M \otimes I + I \otimes M)(V \otimes V) = (M \otimes I)(V \otimes V) + (I \otimes M)(V \otimes V) = MV \otimes V + V \otimes MV = 0$ . Therefore  $\bar{M}(V \otimes V) = 0$  which implies that

$$\dim(\mathcal{N}(\bar{M})) \geq \text{rank}(V \otimes V) = \text{rank}(V) \text{rank}(V) = (n - r)^2$$

Therefore,

$$\text{rank}(\bar{M}) = n^2 - \dim(\mathcal{N}(\bar{M})) \leq n^2 - (n - r)^2 = r(2n - r)$$

■

We are now ready to provide the proof of Theorem 4.1.

*Proof.* [Theorem 4.1]  $(ii) \Rightarrow (i)$ . This is obvious.

$(i) \Rightarrow (ii)$ . First notice that  $\overline{A_0 + \rho A_1} = \bar{A}_0 + \rho \bar{A}_1$ . Since  $A_0 + \rho A_1$  is Hurwitz, according to Lemma 2.2,  $\det(\bar{A}_0 + \rho \bar{A}_1) \neq 0$ . Let  $|\cdot|$  denote the absolute value and let us choose the parameter-dependent, positive definite matrix, given by

$$Q(\rho) = |\det(\bar{A}_0 + \rho \bar{A}_1)|I > 0. \quad (84)$$

Since  $A_0 + \rho A_1$  is Hurwitz for all  $\rho \in \Omega$ , the following Lyapunov equation has a unique positive definite solution  $P(\rho) > 0$  for all  $\rho \in \Omega$  [98]

$$(A_0 + \rho A_1)P(\rho) + P(\rho)(A_0 + \rho A_1)^T + |\det(\bar{A}_0 + \rho \bar{A}_1)|I = 0 \quad (85)$$



Solving this equation, one obtains

$$(\bar{A}_0 + \rho \bar{A}_1) \text{vec}(P) = -|\det(\bar{A}_0 + \rho \bar{A}_1)| \text{vec}(I) \quad (86)$$

and thus

$$\begin{aligned} \text{vec}(P) &= -|\det(\bar{A}_0 + \rho \bar{A}_1)| (\bar{A}_0 + \rho \bar{A}_1)^{-1} \text{vec}(I) \\ &= -|\det(\bar{A}_0 + \rho \bar{A}_1)| \frac{1}{\det(\bar{A}_0 + \rho \bar{A}_1)} \text{Adj}(\bar{A}_0 + \rho \bar{A}_1) \text{vec}(I) \\ &= \sigma(\rho) \text{Adj}(\bar{A}_0 + \rho \bar{A}_1) \text{vec}(I), \end{aligned} \quad (87)$$

where  $\sigma(\rho) = -\text{sign}(\det(\bar{A}_0 + \rho \bar{A}_1))$  and the definition of the  $\text{vec}(\cdot)$  is standard as in [18, 98].

Let  $\bar{r} = \text{rank}(\bar{A}_1)$ . Then according to Lemma 4.2,  $\text{Adj}(\bar{A}_0 + \rho \bar{A}_1)$  is a polynomial of  $\rho$  with degree  $\min\{\bar{r}, n^2 - 1\}$ , i.e.

$$\text{Adj}(\bar{A}_0 + \rho \bar{A}_1) = \sum_{i=0}^{\min\{\bar{r}, n^2-1\}} \rho^i N_i$$

where  $N_i$  are some constant matrices which depend only on  $A_0$  and  $A_1$ . According to Lemma 4.4,  $\bar{r} \leq r(2n - r)$ . Moreover,  $2nr - r^2 \leq n^2 - 1$  for all  $r \leq n - 1$ . Together with (162), one has

$$P = \sigma(\rho)(P_0 + \rho P_1 + \rho^2 P_2 + \dots + \rho^{\bar{r}} P_{\bar{r}}) \quad (88)$$

where  $P_i$ ,  $0 \leq i \leq \bar{r}$  are constant matrices and

$$P_i = \text{vec}^{-1}(N_i \text{vec}(I)), \quad i = 0, 1, 2, \dots, \bar{r} \quad (89)$$

where  $\text{vec}^{-1}(\cdot)$  is the inverse mapping of  $\text{vec}(\cdot)$  and

$$\begin{aligned} N_0 &= \text{Adj}(\bar{A}_0), \\ N_i &= \Gamma_{n^2-1}^i \text{Adj}(\bar{A}_0 / \bar{A}_1^i), \quad i = 1, 2, \dots, \bar{r} \end{aligned}$$

where,  $N_i$ ,  $i = 1, 2, \dots, \bar{r}$  are computed according to Corollary 2.2. ■

**Remark 4.1** Note that if the domain  $\Omega$  is connected then  $\sigma(\rho)$  is constant via Lemma 2.2 and the Lyapunov matrix (88) is given simply by  $P(\rho) = \sum_{i=0}^{\bar{r}} \rho^i P_i$  for all  $\rho \in \Omega$ .

### 4.2.3 Lyapunov Polynomial Function of Reduced Degree

Since only  $\frac{1}{2}n(n+1)$  elements in the Lyapunov matrix  $P = P^T \in \mathbb{R}^{n \times n}$  are independent, the expression of  $\text{vec}(P) \in \mathbb{R}^{n^2}$  is redundant if used to solve for  $P$  from (162). The following vector-valued function is composed of only the independent elements in the matrix  $P$ .

**Definition 4.1** ([53]) *Given a symmetric matrix  $P = P^T \in \mathbb{R}^{n \times n}$ , define*

$$\overline{\text{vec}}(P) := \begin{bmatrix} P_{11} \\ \vdots \\ P_{n1} \\ P_{22} \\ \vdots \\ P_{n2} \\ \vdots \\ P_{nn} \end{bmatrix} \in \mathbb{R}^{\frac{1}{2}n(n+1)} \quad (90)$$

Note that the usual  $\text{vec}(P)$  operator [18] that stacks the columns of a matrix  $P$  one on top of the other consists of all the elements of  $\overline{\text{vec}}(P)$  with some repetitions. For every symmetric matrix  $P = P^T \in \mathbb{R}^{n \times n}$ , there exists a unique full column rank matrix  $D_n \in \mathbb{R}^{n^2 \times \frac{1}{2}n(n+1)}$  called *the duplication matrix* [51, 53], which is independent of the matrix  $P$  and which depends only on the dimension  $n$  of the matrix  $P$ , and which satisfies

$$\text{vec}(P) = D_n \overline{\text{vec}}(P). \quad (91)$$

The pseudo-inverse of  $D_n$  satisfies the following properties [51, 53]

$$\overline{\text{vec}}(P) = D_n^+ \text{vec}(P), \quad D_n^+ D_n = I_{\frac{1}{2}n(n+1)}, \quad \text{rank}(D_n) = \text{rank}(D_n^+) = \frac{1}{2}n(n+1).$$

Notice, in particular, that  $D_n$  is always full column rank. Consequently,  $D_n^+ = (D_n^T D_n)^{-1} D_n^T$ .

**Definition 4.2** ([53]) *Given  $A \in \mathbb{R}^{n \times n}$ , let  $\hat{A} \in \mathbb{R}^{\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)}$  be defined by*

$$\hat{A} := D_n^+ (A \oplus A) D_n = D_n^+ \bar{A} D_n. \quad (92)$$

where  $\bar{A} := A \oplus A = I_n \otimes A + A \otimes I_n$  is the Kronecker sum of matrix  $A$  with itself.

The matrix  $\hat{A}$  is often called the lower Schlaeflian form of or the power form of the matrix  $A$ . It is clear from the definition that  $\hat{A}(\rho) = \widehat{A_0 + \rho A_1} = \hat{A}_0 + \rho \hat{A}_1$ .

The main result in this chapter can be stated as follows:

**Theorem 4.2** *Given the matrices  $A_0, A_1 \in \mathbb{R}^{n \times n}$  with  $\text{rank } A_1 = r$ , let*

$$m := \begin{cases} \frac{1}{2}(2nr - r^2 + r), & \text{if } r < n, \\ \frac{1}{2}n(n+1) - 1, & \text{if } r = n. \end{cases} \quad (93)$$

*Then the following two statements are equivalent:*

(i)  $A_0 + \rho A_1$  is Hurwitz for all  $\rho \in \Omega$ .

(ii) There exists a set of  $m+1$  matrices  $\{P_i\}_{0 \leq i \leq m}$ , such that

$$(A_0 + \rho A_1)^T P(\rho) + P(\rho)(A_0 + \rho A_1) < 0, \quad \forall \rho \in \Omega \quad (94)$$

$$P(\rho) = \sigma(\rho) \left( \sum_{i=0}^m \rho^i P_i \right) > 0, \quad \forall \rho \in \Omega \quad (95)$$

where  $\sigma(\rho) = -\text{sign}(\det(\hat{A}_0 + \rho \hat{A}_1))$ .

**Remark 4.2** Note that if the domain  $\Omega$  is connected then  $\sigma(\rho)$  is constant via Corollary 4.1 (see below) and the Lyapunov matrix (95) is given simply by  $P(\rho) = \sum_{i=0}^m \rho^i P_i$  for all  $\rho \in \Omega$ .

In order to provide the proof of Theorem 4.2 it is needed first to introduce several mathematical preliminaries.

**Lemma 4.5** ([51, 53]) *Given  $A \in \mathbb{R}^{n \times n}$  and  $\hat{A}$  as in Definition 4.2, the eigenvalues of  $\hat{A}$  are the  $\frac{1}{2}n(n+1)$  numbers  $\lambda_i + \lambda_j$ , ( $1 \leq j \leq i \leq n$ ) where  $\lambda_i, \lambda_j$  are the eigenvalues of  $A$ .*

The following is immediate from Lemma 4.5.

**Corollary 4.1** *Suppose the parameter-dependent matrix  $A_0 + \rho A_1 \in \mathbb{R}^{n \times n}$  is Hurwitz for all  $\rho \in \Omega$ . Then*

$$\det(\widehat{A_0 + \rho A_1}) = \det(\widehat{A_0} + \rho \widehat{A_1}) \neq 0, \quad \forall \rho \in \Omega \quad (96)$$

Lemma 4.5 shows the eigenvalues of  $\widehat{A}$ , the Schlaeflian form of the matrix  $A$ , are the  $\frac{1}{2}n(n+1)$  numbers of  $\lambda_i + \lambda_j$ ,  $1 \leq j \leq i \leq n$ , where  $\lambda_i, \lambda_j$  are the eigenvalues of  $A$ . This eigenvalue property of  $\widehat{A}$  is a special case of the eigenvalue property of power form  $A_{[p]}$  [88, 6, 19]. For  $A \in \mathbb{R}^{n \times n}$ ,  $p \in \mathbb{Z}_+^0$  and  $p \geq 2$ , let  $m = \binom{n+p-1}{p}$  and  $A_{[p]} \in \mathbb{R}^{m \times m}$ .  $A_{[p]}$  has  $m$  eigenvalues, which are different sums  $\lambda_i + \lambda_j + \dots + \lambda_k$  ( $p$  terms),  $i, j, k = 1, 2, \dots, n$ . Therefore if  $A(\rho) = A_0 + \rho A_1 \in \mathbb{R}^{n \times n}$  is Hurwitz for  $\rho \in \Omega \subseteq \mathbb{R}$ ,  $A_{[p]}$  is also Hurwitz for  $\rho \in \Omega$  and furthermore  $\det(A_{[p]}) \neq 0$  for  $\rho \in \Omega$ . However, the power form matrix  $A_{[p]}$  will not be used to study the stability of the parameter-dependent LTI systems in this thesis, and not be applied to find the parameter-dependent Lyapunov functions since the size of  $A_{[p]}$ , i.e.,  $m \times m$  is larger or equivalent to the size of  $\widehat{A}$ ,  $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$ ,

$$m = \binom{n+p-1}{p} \geq \frac{1}{2}n(n+1) \quad \forall p \geq 2$$

The following lemma provides a bound for the rank of the Schlaeflian form of a matrix.

**Lemma 4.6** *Given a matrix  $A \in \mathbb{R}^{n \times n}$  with  $\text{rank } A = r$ , then  $\text{rank } \widehat{A} \leq \frac{1}{2}(2nr - r^2 + r)$ .*

The proof of this lemma depends on the special structure of the duplication matrix  $D_n$ . In particular, using Definition 4.1 one can show the following.

**Lemma 4.7** *For any positive integer  $r < n$ , the matrix  $D_n$  has the form*

$$D_n = \begin{bmatrix} D_{11} & 0_{rn \times \frac{1}{2}(n-r)(n-r+1)} \\ D_{21} & D_{22} \end{bmatrix} \quad (97)$$

where  $D_{11} \in \mathbb{R}^{rn \times \frac{1}{2}(2nr - r^2 + r)}$ ,  $D_{21} \in \mathbb{R}^{(n^2 - rn) \times \frac{1}{2}(2nr - r^2 + r)}$  and  $D_{22} \in \mathbb{R}^{(n^2 - rn) \times \frac{1}{2}(n-r)(n-r+1)}$ .

*Proof.* It follows directly from (91) and the fact that the matrix  $P$  is symmetric. Specifically, the first  $rn$  elements of the vector  $\text{vec}(P)$  (which are exactly the first  $r$  columns of  $P$ ), are also the first  $n + (n-1) + \dots + (n-r+1)$  elements of  $\overline{\text{vec}}(P)$  with some repetitions. Therefore,  $D_{12} = 0$ . Moreover, the dimension of  $D_{12}$  is  $rn \times (\frac{1}{2}n(n+1) - (n + (n+1) + \dots + (n-r+1))) = rn \times \frac{1}{2}(n-r)(n-r+1)$ . ■

The proof of the following properties of the duplication matrix can be found in [51].

**Lemma 4.8 ([51])** *Given a matrices  $A, B \in \mathbb{R}^{n \times n}$  the following hold:*

- (i)  $D_n^+(A \otimes B)D_n = D_n^+(B \otimes A)D_n = \frac{1}{2}D_n^+(B \otimes A + A \otimes B)D_n$ .
- (ii)  $D_n D_n^+(I_n \otimes A + A \otimes I_n)D_n = (I_n \otimes A + A \otimes I_n)D_n$ .
- (iii)  $(D_n^+(A \otimes A)D_n)^{-1} = D_n^+(A^{-1} \otimes A^{-1})D_n$ .
- (iv)  $D_n D_n^+(A \otimes A)D_n = (A \otimes A)D_n$ .

*Proof.* [of Lemma 4.6] Since  $\text{rank } A = r$ , it follows that there exist two nonsingular matrices  $U, V \in \mathbb{R}^{n \times n}$  such that

$$UAV = \begin{bmatrix} M & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix} \quad (98)$$

where  $M \in \mathbb{R}^{r \times r}$ . Since  $D_n^+(U \otimes U)D_n$  and  $D_n^+(V \otimes V)D_n$  are nonsingular from property (iii) of Lemma 4.8, and there is

$$\text{rank } \hat{A} = \text{rank} \left( D_n^+(U \otimes U)D_n \hat{A} D_n^+(V \otimes V)D_n \right), \quad (99)$$

due to the property that  $\text{rank } A = \text{rank}(BAC)$  for any nonsingular matrices  $B$  and  $C$  [38].

Notice now that

$$\begin{aligned}
D_n^+(U \otimes U)D_n\widehat{A}D_n^+(V \otimes V)D_n &= D_n^+(U \otimes U)D_nD_n^+(I_n \otimes A + A \otimes I_n)D_nD_n^+(V \otimes V)D_n \\
&= D_n^+(U \otimes U)(I_n \otimes A + A \otimes I_n)D_nD_n^+(V \otimes V)D_n \\
&= D_n^+(U \otimes U)(I_n \otimes A + A \otimes I_n)(V \otimes V)D_n \\
&= D_n^+\left(U \otimes (UA) + (UA) \otimes U\right)(V \otimes V)D_n \\
&= D_n^+\left((UV) \otimes (UAV) + (UAV) \otimes (UV)\right)D_n \\
&= 2D_n^+\left((UAV) \otimes (UV)\right)D_n
\end{aligned} \tag{100}$$

Using (98), one obtains

$$\begin{aligned}
D_n^+(U \otimes U)D_n\widehat{A}D_n^+(V \otimes V)D_n &= 2D_n^+ \begin{bmatrix} M \otimes (UV) & 0_{r \times (n-r)} \otimes (UV) \\ 0_{(n-r) \times r} \otimes (UV) & 0_{(n-r) \times (n-r)} \otimes (UV) \end{bmatrix} D_n \\
&= 2D_n^+ \begin{bmatrix} L & 0_{rn \times n(n-r)} \\ 0_{n(n-r) \times rn} & 0_{n(n-r) \times n(n-r)} \end{bmatrix} D_n
\end{aligned}$$

where  $L := M \otimes (UV) \in \mathbb{R}^{rn \times rn}$ . Therefore, from (97)

$$\begin{aligned}
\text{rank } \widehat{A} &= \text{rank} \left( D_n^+ \begin{bmatrix} L & 0_{rn \times n(n-r)} \\ 0_{n(n-r) \times rn} & 0_{n(n-r) \times n(n-r)} \end{bmatrix} \begin{bmatrix} D_{11} & 0_{rn \times \frac{1}{2}(n-r)(n-r+1)} \\ D_{21} & D_{22} \end{bmatrix} \right) \\
&= \text{rank} \left( D_n^+ \begin{bmatrix} LD_{11} & 0_{rn \times \frac{1}{2}(n-r)(n-r+1)} \\ 0_{n(n-r) \times \frac{1}{2}(2nr-r^2+r)} & 0_{n(n-r) \times \frac{1}{2}(n-r)(n-r+1)} \end{bmatrix} \right) \\
&\leq \text{rank } D_{11}
\end{aligned} \tag{101}$$

Since  $D_{11} \in \mathbb{R}^{rn \times \frac{1}{2}(2nr-r^2+r)}$ , and since  $rn \geq \frac{1}{2}(2nr-r^2+r)$  for all  $r \geq 1$ , it follows that  $\text{rank } D_{11} \leq \frac{1}{2}(2nr-r^2+r)$ . Finally,  $\text{rank } \widehat{A} \leq \frac{1}{2}(2nr-r^2+r)$ .  $\blacksquare$

It is now ready to provide the proof of Theorem 4.2.

*Proof.* [Of Theorem 4.2] (ii)  $\Rightarrow$  (i): This is obvious.

(i)  $\Rightarrow$  (ii): Since  $A_0 + \rho A_1$  is Hurwitz for all  $\rho \in \Omega$ , from Corollary 4.1 it follows that  $\det(\widehat{A}_0 + \rho \widehat{A}_1) \neq 0$ . Let the parameter-dependent matrix

$$Q(\rho) := |\det(\widehat{A}_0 + \rho \widehat{A}_1)|I_n > 0, \quad \rho \in \Omega. \tag{102}$$

Note that  $Q(\rho)$  is positive definite for each  $\rho \in \Omega$ . Since  $A_0 + \rho A_1$  is Hurwitz for all  $\rho \in \Omega$ , the following Lyapunov equation has a unique, positive definite-solution  $P(\rho) > 0$  for each  $\rho \in \Omega$

$$(A_0 + \rho A_1)P(\rho) + P(\rho)(A_0 + \rho A_1)^T + |\det(\hat{A}_0 + \rho \hat{A}_1)|I_n = 0. \quad (103)$$

Solving this equation for  $P(\rho)$  one obtains

$$\begin{aligned} ((A_0 + \rho A_1) \oplus (A_0 + \rho A_1))\text{vec}(P(\rho)) &= -|\det(\hat{A}_0 + \rho \hat{A}_1)|\text{vec}(I_n) \\ \overline{(A_0 + \rho A_1)}\text{vec}(P(\rho)) &= -|\det(\hat{A}_0 + \rho \hat{A}_1)|\text{vec}(I_n) \\ (\bar{A}_0 + \rho \bar{A}_1)\text{vec}(P(\rho)) &= -|\det(\hat{A}_0 + \rho \hat{A}_1)|\text{vec}(I_n) \\ D_n^+(\bar{A}_0 + \rho \bar{A}_1)D_n\overline{\text{vec}}(P(\rho)) &= -|\det(\hat{A}_0 + \rho \hat{A}_1)|\overline{\text{vec}}(I_n) \\ (\hat{A}_0 + \rho \hat{A}_1)\overline{\text{vec}}(P(\rho)) &= -|\det(\hat{A}_0 + \rho \hat{A}_1)|\overline{\text{vec}}(I_n) \\ \overline{\text{vec}}(P(\rho)) &= -|\det(\hat{A}_0 + \rho \hat{A}_1)|(\hat{A}_0 + \rho \hat{A}_1)^{-1}\overline{\text{vec}}(I_n) \end{aligned}$$

and thus,

$$\begin{aligned} \overline{\text{vec}}(P(\rho)) &= -|\det(\hat{A}_0 + \rho \hat{A}_1)|\frac{\text{Adj}(\hat{A}_0 + \rho \hat{A}_1)}{\det(\hat{A}_0 + \rho \hat{A}_1)}\overline{\text{vec}}(I_n) \\ &= \sigma(\rho)\text{Adj}(\hat{A}_0 + \rho \hat{A}_1)\overline{\text{vec}}(I_n) \end{aligned} \quad (104)$$

where  $\sigma(\rho) := -\text{sign}(\det(\hat{A}_0 + \rho \hat{A}_1))$ .

Let  $\hat{r} := \text{rank } \hat{A}_1$ . According to Lemma 4.6 there exists  $\hat{r} \leq \frac{1}{2}(2nr - r^2 + r)$ . Moreover, according to Lemma 4.2 there exist constant matrices  $N_i$  such that  $\text{Adj}(\hat{A}_0 + \rho \hat{A}_1) = \sum_{i=0}^m \rho^i N_i$  where  $m = \min\{\hat{r}, \frac{1}{2}n(n+1) - 1\} \leq \min\{\frac{1}{2}(2nr - r^2 + r), \frac{1}{2}n(n+1) - 1\}$ . Notice, in particular that  $\min\{\frac{1}{2}(2nr - r^2 + r), \frac{1}{2}n(n+1) - 1\} = \frac{1}{2}(2nr - r^2 + r)$  for  $r < n$  and  $\min\{\frac{1}{2}(2nr - r^2 + r), \frac{1}{2}n(n+1) - 1\} = \frac{1}{2}n(n+1) - 1$  if  $r = n$ .

Moreover, since the mapping  $\overline{\text{vec}}(\cdot)$  is one-to-one, its inverse mapping  $\overline{\text{vec}}^{-1}(\cdot)$  exists. Therefore, (104) finally yields

$$P(\rho) = \sigma(\rho) \left( \sum_{i=0}^m \rho^i P_i \right) \quad (105)$$

where  $P_i \in \mathbb{R}^{n \times n}$  are constant matrices given by  $P_i = \overline{\text{vec}}^{-1}\left(N_i \overline{\text{vec}}(I_n)\right)$  for  $0 \leq i \leq m$ , and

$$\begin{aligned} N_0 &= \text{Adj}(\widehat{A}_0), \\ N_i &= \Gamma_{n^2-1}^i \text{Adj}(\widehat{A}_0 / \widehat{A}_1^i), \quad i = 1, 2, \dots, m \end{aligned}$$

where,  $N_i$ ,  $i = 0, 1, 2, \dots, m$  can be computed according to Corollary 2.2. ■

**Remark 4.3** Note that if the domain  $\Omega$  is connected then  $\sigma(\rho)$  is constant via Corollary 4.1 and the Lyapunov matrix (95) is given simply by  $P(\rho) = \sum_{i=0}^m \rho^i P_i$  for all  $\rho \in \Omega$ .

#### 4.2.4 Numerical Examples

The parameter-dependent system matrices  $A(\rho)$  in the following examples are taken from [94]. In these examples, the parameter-dependent Lyapunov matrix  $P(\rho)$  is calculated and the corresponding Lyapunov inequalities are checked. It is reminded that the exact stability domain may be a single interval or a union of a finite number of disjointed intervals [94].

**Example 4.3** Consider the matrix  $A(\rho) = A_0 + \rho A_1$ , where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The largest stability domain of  $A(\rho)$  for this example is  $(-\infty, 2)$ . To compute  $P(\rho)$ , first notice that

$$\widehat{A}_0 = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \quad \widehat{A}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and  $\det(\widehat{A}_0 + \rho \widehat{A}_1) = (-4 + 2\rho)^3 = -64 + 96\rho - 48\rho^2 + 8\rho^3$ . Thus,

$$\text{Adj}(\widehat{A}_0 + \rho \widehat{A}_1) = \begin{bmatrix} 4\rho^2 - 16\rho + 16 & 0 & 0 \\ 0 & 4\rho^2 - 16\rho + 16 & 0 \\ 0 & 0 & 4\rho^2 - 16\rho + 16 \end{bmatrix},$$



and the parameter-dependent Lyapunov matrix is given by

$$P(\rho) = \begin{bmatrix} 4\rho^2 - 16\rho + 16 & 0 \\ 0 & 4\rho^2 - 16\rho + 16 \end{bmatrix}.$$

Moreover,

$$A(\rho)P(\rho) + P(\rho)A^T(\rho) = \begin{bmatrix} -64 + 96\rho - 48\rho^2 + 8\rho^3 & 0 \\ 0 & -64 + 96\rho - 48\rho^2 + 8\rho^3 \end{bmatrix}.$$

The eigenvalues of  $P(\rho)$  are given by  $\lambda_{1,2} = 4\rho^2 - 16\rho + 16 = (2\rho - 4)^2$ . Note that  $\lambda_{1,2}(\rho) > 0$  if and only if  $\rho \neq 2$  and therefore  $P(\rho) > 0$  if and only if  $\rho \neq 2$ . However, only for  $\rho \in (-\infty, 2)$ , it holds that  $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$ .

**Example 4.4** Consider the matrix  $A(\rho) = A_0 + \rho A_1$ , where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The largest stability domain for this example is  $(-1, 2)$ . To compute  $P(\rho)$ , first note that

$$\hat{A}_0 = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \hat{A}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and  $\det(\hat{A}_0 + \rho\hat{A}_1) = (-4 + 2\rho)(6 + 6\rho) = -24 - 12\rho + 12\rho^2$ . Therefore,

$$\text{Adj}(\hat{A}_0 + \rho\hat{A}_1) = \begin{bmatrix} 6\rho + 6 & 0 & 0 \\ 0 & -4\rho^2 + 4\rho + 8 & 0 \\ 0 & 0 & -6\rho + 12 \end{bmatrix},$$

and thus,

$$P(\rho) = \begin{bmatrix} 6\rho + 6 & 0 \\ 0 & -6\rho + 12 \end{bmatrix}.$$

Moreover,

$$A(\rho)P(\rho) + P(\rho)A^T(\rho) = \begin{bmatrix} -24 - 12\rho + 12\rho^2 & 0 \\ 0 & -24 - 12\rho + 12\rho^2 \end{bmatrix}.$$

The eigenvalues of  $P(\rho)$  are given by  $\lambda_1 = 6\rho + 6$  and  $\lambda_2 = -6\rho + 12$ . Notice that  $\lambda_{1,2}(\rho) > 0$  for  $\rho \in (-1, 2)$  and therefore  $P(\rho) > 0$  for  $\rho \in (-1, 2)$ . Furthermore,  $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$  for  $\rho \in (-1, 2)$ .

**Example 4.5** Consider the parameter-dependent matrix  $A(\rho) = A_0 + \rho A_1$ , where

$$A_0 = \begin{bmatrix} 0.7493 & -2.4358 & -1.6503 \\ -2.0590 & -3.3003 & -1.4833 \\ -1.5019 & 1.2149 & -4.8737 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1.2149 & 1.6640 & -2.2091 \\ 0.7542 & -0.1501 & 0.2109 \\ 2.1990 & 0.6493 & -0.2214 \end{bmatrix} \quad (106)$$

The exact stability domain for  $A(\rho)$  is  $(-18.3861, -1.2729) \cup (2.1538, 3.7973)$ , which is computed with the method presented in [94]. With the method introduced in the proof of Theorem 4.2, one can first compute  $\hat{A}_0, \hat{A}_1$  and then  $\text{Adj}(\hat{A}_0 + \rho \hat{A}_1)$ . The matrix  $P(\rho)$ , is a polynomial in  $\rho$  of degree  $m = \frac{1}{2}n(n+1) - 1 = 5$  since  $r = \text{rank}(A_1) = n = 3$ .

In this example the stability domain is composed of two disjoint intervals. The parameter-dependent Lyapunov function is given by

$$P(\rho) = \sigma(\rho) \left( P_0 + \rho P_1 + \rho^2 P_2 + \rho^3 P_3 + \rho^4 P_4 + \rho^5 P_5 \right)$$

where,

$$\begin{aligned} P_0 &= \begin{bmatrix} 7.4544 & -1.3754 & -2.8700 \\ -1.3754 & -3.6225 & 0.7518 \\ -2.8700 & 0.7518 & -1.7334 \end{bmatrix}, & P_1 &= \begin{bmatrix} -4.1133 & -2.8221 & 7.6811 \\ -2.8221 & 1.0147 & -1.4137 \\ 7.6811 & -1.4137 & -4.7021 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 5.4508 & -4.2469 & -5.3219 \\ -4.2469 & 2.3203 & 1.3195 \\ -5.3219 & 1.3195 & 6.1676 \end{bmatrix}, & P_3 &= \begin{bmatrix} -1.4747 & 2.7044 & 2.1666 \\ 2.7044 & -1.7968 & -1.9922 \\ 2.1666 & -1.9922 & -3.6788 \end{bmatrix}, \\ P_4 &= \begin{bmatrix} 0.7450 & -0.3384 & 0.0541 \\ -0.3384 & 0.6999 & 0.7148 \\ 0.0541 & 0.7148 & 1.2017 \end{bmatrix}, & P_5 &= \begin{bmatrix} 0.03532 & 0.1186 & 0.02280 \\ 0.01186 & -0.10092 & -0.05606 \\ 0.02280 & -0.05606 & 0.06675 \end{bmatrix} \end{aligned}$$

It can be numerically checked that  $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$  for all  $\rho \in \mathbb{R}$ . However,  $P(\rho)$  is positive definite only for  $\rho \in (-18.3861, -1.2729) \cup (2.1538, 3.7973)$ .

**Remark 4.4** Theorem 4.2 can be used to determine the whole stability domain of a parameter-dependent LTI system, even if this domain is composed of several disjoint intervals of  $\mathbb{R}$  as is the case in Example 4.5. The approach of [7, 54, 53, 24, 64, 65] without modification, on the other hand, can only be used to check the stability over a connected domain which includes the origin.

### 4.3 *Convex Characterization of the Matrix Inequalities*

The previous analysis shows that the parameter-dependent matrix  $A(\rho) = A_0 + \rho A_1$  is Hurwitz for any  $\rho \in \Omega$ , if and only if there exists a Lyapunov matrix which depends polynomially on the parameter  $\rho$ , of the form

$$P(\rho) := P_0 + \rho P_1 + \dots + \rho^m P_m, \quad (107)$$

such that the corresponding two matrix inequalities

$$A(\rho)P(\rho) + P(\rho)A(\rho)^T < 0, \quad (108)$$

$$P(\rho) > 0, \quad (109)$$

are satisfied. In order to be able to use the stability criterion of Theorem 4.2 in practice, it is desired to have a relatively simple method to determine the feasibility of the matrix inequalities (108) and (109).

This section presents computable, non-conservative, conditions to test the matrix inequalities (108) and (109) over any compact interval  $\Omega$ . Without loss of generality, in the sequel it is assumed that  $\Omega := [-1, 1]$ .

To this end, let the vector  $\rho^{[q]} \in \mathbb{R}^q$  be defined by

**Definition 4.3**

$$\rho^{[q]} := \begin{pmatrix} 1 \\ \rho \\ \rho^2 \\ \vdots \\ \rho^{q-1} \end{pmatrix}, \quad (110)$$

and notice that the parameter-dependent matrix in (107) can be rewritten as

$$P(\rho) = \left( \rho^{[k]} \otimes I_n \right)^T P_\Sigma \left( \rho^{[k]} \otimes I_n \right) \quad (111)$$

where  $k = \lceil \frac{m}{2} \rceil + 1$  and  $P_\Sigma = P_\Sigma^T \in \mathbb{R}^{nk \times nk}$  is a constant symmetric matrix (here  $\lceil \frac{m}{2} \rceil$  denotes the smallest integer which is larger than or equal to  $m/2$ ). Note that the matrix  $P_\Sigma$  is not unique. One possible choice is given by

$$P_\Sigma := \frac{1}{2} \begin{bmatrix} 2P_0 & P_1 & & & \\ P_1 & 2P_2 & P_3 & & \\ & P_3 & 2P_4 & \ddots & \\ & & \ddots & \ddots & P_{m-1} \\ & & & P_{m-1} & 2P_m \end{bmatrix}, \quad \text{if } m \text{ is even,}$$

and

$$P_\Sigma := \frac{1}{2} \begin{bmatrix} 2P_0 & P_1 & & & & \\ P_1 & 2P_2 & P_3 & & & \\ & P_3 & 2P_4 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & P_{m-2} & 2P_{m-1} & P_m \\ & & & & P_m & 0 \end{bmatrix}, \quad \text{if } m \text{ is odd.}$$

On the other hand, for any given symmetric matrix  $P_\Sigma$  one can also get a unique polynomial Lyapunov matrix  $P(\rho)$  in the form (107) using the expression (111).

The following lemma provides a convenient expression for the matrix  $R(\rho) = A(\rho)P(\rho) + P(\rho)A^T(\rho)$  which will be useful for providing a convex characterization of inequality (108).

**Lemma 4.9 ([13])** *Given a matrix  $A(\rho) = A_0 + \rho A_1 \in \mathbb{R}^{n \times n}$  and a symmetric, parameter-dependent matrix  $P(\rho) \in \mathbb{R}^{n \times n}$  as*

$$P(\rho) = (\rho^{[k]} \otimes I_n)^T P_\Sigma (\rho^{[k]} \otimes I_n),$$

*let  $R(\rho) := A^T(\rho)P(\rho) + P(\rho)A(\rho)$ . Then*

$$R(\rho) = (\rho^{[k+1]} \otimes I_n)^T R_\Sigma (\rho^{[k+1]} \otimes I_n) \quad (112)$$

*where,*

$$R_\Sigma = H_\Sigma^T P_\Sigma F_\Sigma + F_\Sigma^T P_\Sigma H_\Sigma \quad (113)$$

$$H_\Sigma = \hat{J}_k \otimes I_n \quad (114)$$

$$F_\Sigma = \hat{J}_k \otimes A_0 + \check{J}_k \otimes A_1 \quad (115)$$

$$\text{and } \hat{J}_k := \begin{bmatrix} I_k & 0_{k \times 1} \end{bmatrix} \text{ and } \check{J}_k := \begin{bmatrix} 0_{k \times 1} & I_k \end{bmatrix}.$$

Notice that the matrix  $R_\Sigma$  depends linearly upon each of the matrices  $P_\Sigma$ ,  $A_0$  and  $A_1$ .

The following lemma is instrumental in casting the matrix feasibility problem (108)-(109) to a convex optimization problem. It is an extension of a result given in [39].

**Lemma 4.10** *Let the matrices  $\Theta = \Theta^T \in \mathbb{R}^{n \times n}$  and  $J, C \in \mathbb{R}^{k \times n}$  be given. The following statements are equivalent.*

(i) *The inequality  $\zeta^T \Theta \zeta < 0$  holds for all nonzero vectors  $\zeta \in \mathbb{R}^n$  which satisfy  $(J - \delta C)\zeta = 0$ , for some real scalar  $\delta$  such that  $|\delta| \leq 1$ .*

(ii) *There exist matrices  $D \in \mathbb{R}^{k \times k}$  and  $G \in \mathbb{R}^{k \times k}$  such that*

$$D = D^T > 0, \quad G + G^T = 0, \quad \Theta < \begin{bmatrix} C \\ J \end{bmatrix}^T \begin{bmatrix} -D & G \\ G^T & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix}. \quad (116)$$

*Proof.* From Lemma 2 of [39], a vector  $\zeta \in \mathbb{C}^n$  satisfies  $(J - j\delta(jC))\zeta = 0$  for some real scalar  $\delta$  such that  $|\delta| \leq 1$  if and only if

$$\zeta^* \begin{bmatrix} J \\ jC \end{bmatrix}^* \begin{bmatrix} U & V \\ V & -U \end{bmatrix} \begin{bmatrix} J \\ jC \end{bmatrix} \zeta \leq 0$$

holds for all  $U = U^* > 0$  and  $V = V^*$ . Then applying the generalized  $\mathcal{S}$ -procedure ([39], Theorem 1) and noting the losslessness of the following set of Hermitian matrices

$$\mathcal{S} := \left\{ \begin{bmatrix} J \\ jC \end{bmatrix}^* \begin{bmatrix} U & V \\ V & -U \end{bmatrix} \begin{bmatrix} J \\ jC \end{bmatrix} : V = V^*, U = U^* > 0 \right\}$$

we see ([39], Lemma 3) that statement (i) is equivalent to the existence of a matrix  $V = V^*$  and  $U = U^* > 0$  such that

$$\Theta < \begin{bmatrix} J \\ jC \end{bmatrix}^* \begin{bmatrix} U & V \\ V & -U \end{bmatrix} \begin{bmatrix} J \\ jC \end{bmatrix}, \quad (117)$$

The result then follows by letting  $D := U$  and  $G := -jV$ , and taking the real part of the equation. ■

#### 4.3.1 LMI Conditions for Checking the Stability of LTIPD Systems

It is desirable to find computable, convex, non-conservative conditions to test the stability conditions (108)-(109). Using (111) and (112), the inequalities (108)-(109) can be rewritten as

$$\left( \rho^{[k]} \otimes I_n \right)^T P_\Sigma \left( \rho^{[k]} \otimes I_n \right) > 0, \quad \forall \rho \in \Omega, \quad (118)$$

$$\left( \rho^{[k+1]} \otimes I_n \right)^T R_\Sigma \left( \rho^{[k+1]} \otimes I_n \right) < 0, \quad \forall \rho \in \Omega. \quad (119)$$

**Lemma 4.11** *Given the matrices  $J = \check{J}_{k-1} \otimes I_n \in \mathbb{R}^{n(k-1) \times nk}$  and  $C = \hat{J}_{k-1} \otimes I_n \in \mathbb{R}^{n(k-1) \times nk}$ , the sets  $\mathcal{C}_1$  and  $\mathcal{C}_2$  below are equal.*

$$\mathcal{C}_1 := \{ \zeta \in \mathbb{R}^{nk} : (J - \delta C)\zeta = 0, \text{ some } \delta \in [-1, +1] \}, \quad (120)$$

$$\mathcal{C}_2 := \{ \zeta \in \mathbb{R}^{nk} : \zeta = (\rho^{[k]} \otimes I_n)z, \rho \in [-1, +1], z \in \mathbb{R}^n \}. \quad (121)$$

*Proof.* Since  $(\check{J}_{k-1} - \rho \hat{J}_{k-1})\rho^{[k]} = 0$  and

$$\begin{aligned}
(J - \rho C)(\rho^{[k]} \otimes I_n)z &= (\check{J}_{k-1} \otimes I_n - \rho \hat{J}_{k-1} \otimes I_n)(\rho^{[k]} \otimes I_n)z \\
&= ((\check{J}_{k-1} - \rho \hat{J}_{k-1}) \otimes I_n)(\rho^{[k]} \otimes I_n)z \\
&= (((\check{J}_{k-1} - \rho \hat{J}_{k-1})\rho^{[k]}) \otimes I_n)z \\
&= (0 \otimes I_n)z = 0,
\end{aligned}$$

it follows immediately that  $\mathcal{C}_2 \subseteq \mathcal{C}_1$ .

Conversely, let  $\zeta \in \mathcal{C}_1$ . Since  $\zeta \in \mathbb{R}^{nk}$ , we may write

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_k \end{bmatrix}, \quad \text{where } \zeta_i \in \mathbb{R}^n, \quad i = 1, 2, \dots, k.$$

Since  $(J - \delta C)\zeta = 0$  for some  $\delta \in [-1, +1]$ , it follows that

$$\left[ \begin{bmatrix} 0_{(k-1),1} \otimes I_n, & I_{k-1} \otimes I_n \end{bmatrix} - \delta \begin{bmatrix} I_{k-1} \otimes I_n, & 0_{(k-1),1} \otimes I_n \end{bmatrix} \right] \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_k \end{bmatrix} = 0$$

and expanding the previous expression componentwise,

$$\zeta_2 - \delta \zeta_1 = 0$$

$$\zeta_3 - \delta \zeta_2 = 0$$

$$\vdots$$

$$\zeta_k - \delta \zeta_{k-1} = 0$$

or

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_k \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \delta \zeta_1 \\ \delta^2 \zeta_1 \\ \vdots \\ \delta^{k-1} \zeta_1 \end{bmatrix} = \delta^{[k]} \otimes \zeta_1.$$

Let now  $z := \zeta_1 \in \mathbb{R}^n$ , and  $\rho := \delta \in [-1, +1]$ . It follows that  $\zeta = (\rho^{[k]} \otimes I_n)z$  and thus  $\zeta \in \mathcal{C}_2$ . Therefore,  $\mathcal{C}_1 \subseteq \mathcal{C}_2$  and the claim is shown.  $\blacksquare$

**Lemma 4.12** *Let  $J := \check{J}_{k-1} \otimes I_n = [0 \ I] \in \mathbb{R}^{n(k-1) \times nk}$  and  $C := \hat{J}_{k-1} \otimes I_n = [I \ 0] \in \mathbb{R}^{n(k-1) \times nk}$ . Then the matrix inequality*

$$\left( \rho^{[k]} \otimes I \right)^T \Theta \left( \rho^{[k]} \otimes I \right) < 0 \quad (122)$$

*holds for all  $\rho \in [-1, 1]$  if and only if there exist matrices  $D \in \mathbb{R}^{n(k-1) \times n(k-1)}$  and  $G \in \mathbb{R}^{n(k-1) \times n(k-1)}$  such that*

$$D = D^T > 0, \quad G + G^T = 0, \quad \Theta < \begin{bmatrix} C \\ J \end{bmatrix}^T \begin{bmatrix} -D & G \\ G^T & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix}. \quad (123)$$

*Proof.* The matrix inequality (122) is equivalent to the condition

$$z^T \left( \rho^{[k]} \otimes I_n \right)^T \Theta \left( \rho^{[k]} \otimes I_n \right) z < 0, \quad \forall \rho \in [-1, 1], \quad \forall z \in \mathbb{R}^n \quad (124)$$

Let  $\zeta = \left( \rho^{[k]} \otimes I_n \right) z$  and notice that  $(J - \rho C)(\rho^{[k]} \otimes I_n) = 0$ . Therefore,  $\zeta$  satisfies the constraint  $(J - \rho C)\zeta = 0$  for all real  $\rho$  such that  $|\rho| \leq 1$ . From Lemma 4.11,  $\mathcal{C}_1$  in (120) and  $\mathcal{C}_2$  in (121) are equivalent. Applying now Lemma 4.10, one has that  $\zeta^T \Theta \zeta < 0$  if and only if there exist matrices  $D \in \mathbb{R}^{n(k-1) \times n(k-1)}$  and  $G \in \mathbb{R}^{n(k-1) \times n(k-1)}$  such that (123) holds.  $\blacksquare$

**Example 4.6** Let  $P(\rho) = (1 + \epsilon)I_n - \rho^2 I_n$ . It is clear that if  $\epsilon > 0$ ,  $P(\rho)$  is positive definite for all  $\rho \in [-1, 1]$ . If, on the other hand,  $\epsilon < 0$ ,  $P(\rho)$  is not positive definite for all  $\rho \in [-1, 1]$ . Rewriting  $P(\rho)$  in the form (111),

$$\begin{aligned} P(\rho) &= \left( \rho^{[2]} \otimes I_n \right)^T P_\Sigma \left( \rho^{[2]} \otimes I_n \right) \\ &= \begin{bmatrix} I_n \\ \rho I_n \end{bmatrix}^T \begin{bmatrix} (1 + \epsilon)I_n & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} I_n \\ \rho I_n \end{bmatrix} \end{aligned} \quad (125)$$



and applying Lemma 4.12, with  $k = 2$ , the condition  $P(\rho) > 0$  for all  $\rho \in [-1, 1]$  is equivalent to the existence of matrices  $D = D^T > 0$  and  $G + G^T = 0$  such that

$$-P_\Sigma < \begin{bmatrix} C \\ J \end{bmatrix}^T \begin{bmatrix} -D & G \\ G^T & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix} \quad (126)$$

where  $J = [0_{n \times n} \ I_n]$  and  $C = [I_n \ 0_{n \times n}]$ . The matrix inequality (126) is equivalent to the existence of matrices  $D = D^T > 0$  and  $G + G^T = 0$  such that

$$\begin{bmatrix} D - (1 + \epsilon)I_n & -G \\ -G^T & I_n - D \end{bmatrix} < 0. \quad (127)$$

A necessary condition for the existence of  $D$  in (127) is  $I_n < D < (1 + \epsilon)I_n$ . When  $\epsilon > 0$ , such a  $D$  exists and along with  $G = 0$  the LMI (126) is feasible. When  $\epsilon < 0$ , no  $D$  can satisfy (127) and the LMI (127) has no solution. For both cases, the result of Lemma 4.12 agrees with the direct stability analysis.

The following is a direct consequence of Lemma 4.12. It provides convex conditions in terms of LMIs for checking the robust stability of the parameter dependent matrix  $A(\rho) = A_0 + \rho A_1$  for  $\rho \in [-1, +1]$ .

**Theorem 4.3** *Let the parameter-dependent matrix  $A(\rho) = A_0 + \rho A_1$ , where  $A_0, A_1 \in \mathbb{R}^{n \times n}$  with  $\text{rank } A_1 = r$  and let  $k = \lceil \frac{m}{2} \rceil + 1$  where*

$$m := \begin{cases} \frac{1}{2}(2nr - r^2 + r), & \text{if } r < n, \\ \frac{1}{2}n(n+1) - 1, & \text{if } r = n. \end{cases} \quad (128)$$

*Let  $J_1 = [0 \ I_{n(k-1)}] \in \mathbb{R}^{n(k-1) \times nk}$ ,  $C_1 = [I_{n(k-1)} \ 0] \in \mathbb{R}^{n(k-1) \times nk}$ ,  $J_2 = [0 \ I_{nk}] \in \mathbb{R}^{nk \times n(k+1)}$  and  $C_2 = [I_{nk} \ 0] \in \mathbb{R}^{nk \times n(k+1)}$ . Then,  $A(\rho)$  is Hurwitz for all  $\rho \in [-1, 1]$  if and only if there exist symmetric matrices  $P_\Sigma \in \mathbb{R}^{nk \times nk}$ ,  $D_1 \in \mathbb{R}^{n(k-1) \times n(k-1)}$  and  $D_2 \in \mathbb{R}^{nk \times nk}$  and*

skew-symmetric matrices  $G_1 \in \mathbb{R}^{n(k-1) \times n(k-1)}$ ,  $G_2 \in \mathbb{R}^{nk \times nk}$ , such that

$$D_1 = D_1^T > 0, \quad G_1 + G_1^T = 0, \quad -P_\Sigma < \begin{bmatrix} C_1 \\ J_1 \end{bmatrix}^T \begin{bmatrix} -D_1 & G_1 \\ G_1^T & D_1 \end{bmatrix} \begin{bmatrix} C_1 \\ J_1 \end{bmatrix}, \quad (129)$$

$$D_2 = D_2^T > 0, \quad G_2 + G_2^T = 0, \quad R_\Sigma < \begin{bmatrix} C_2 \\ J_2 \end{bmatrix}^T \begin{bmatrix} -D_2 & G_2 \\ G_2^T & D_2 \end{bmatrix} \begin{bmatrix} C_2 \\ J_2 \end{bmatrix}, \quad (130)$$

where  $R_\Sigma = R_\Sigma(P_\Sigma)$  as in (113)-(115).

*Proof.* According to Theorem 4.2,  $A(\rho)$  is Hurwitz if and only if there exists a matrix  $P(\rho)$  which depends polynomially on the parameter  $\rho$  of degree  $m$  as in (107), such that the matrix inequalities (108) and (109) are satisfied. From Lemma 4.9 these inequalities can be written in the form (118) and (119). Lemma 4.12 shows that the inequalities (118) and (119) are equivalent to the feasibility of the LMI conditions (129) and (130), respectively. ■

**Example 4.7** Let  $A(\rho) = -(1 + \epsilon)I_2 + \rho I_2$ . Here  $A_0 = -(1 + \epsilon)I_2$  and  $A_1 = I_2$ . It is clear that if  $\epsilon > 0$ ,  $A(\rho)$  is Hurwitz for all  $\rho \in [-1, 1]$  whereas if  $\epsilon < 0$ ,  $A(\rho)$  is not Hurwitz for all  $\rho \in [-1, 1]$ . Applying Theorem 4.3 with  $n = 2$  and  $m = \frac{1}{2}n(n+1) - 1 = 2$  one has

$$\begin{aligned} P(\rho) &= P_0 + \rho P_1 + \rho^2 P_2 = \left( \rho^{[2]} \otimes I_2 \right)^T P_\Sigma \left( \rho^{[2]} \otimes I_2 \right) \\ R(\rho) &= A^T(\rho)P(\rho) + P(\rho)A(\rho) = \left( \rho^{[3]} \otimes I_2 \right)^T R_\Sigma \left( \rho^{[3]} \otimes I_2 \right) \end{aligned}$$

where,

$$P_\Sigma = \begin{bmatrix} P_0 & 0.5P_1 \\ 0.5P_1 & P_2 \end{bmatrix} \quad (131)$$

and

$$R_\Sigma = \begin{bmatrix} (A_0^T P_0 + P_0 A_0) & \star & \star \\ 0.5(A_0^T P_1 + P_1 A_0 + P_0 A_1 + A_1^T P_0) & (P_2 A_0 + A_0^T P_2 + P_1 A_1 + A_1^T P_1) & \star \\ 0 & 0.5(P_2 A_1 + A_1^T P_2) & 0 \end{bmatrix}. \quad (132)$$

Let  $J_1 = [0_{2 \times 2} \ I_2]$ ,  $C_1 = [I_2 \ 0_{2 \times 2}]$ ,  $J_2 = [0_{4 \times 2} \ I_4]$  and  $C_2 = [I_4 \ 0_{4 \times 2}]$  as in Theorem 4.3. Using the MATLAB<sup>TM</sup> LMI Toolbox [28] one can solve (129) and (130) with  $\epsilon = 0.001$  to obtain the solution

$$P_0 = \begin{bmatrix} 3.804 & 0 \\ 0 & 3.804 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1.137 & 0 \\ 0 & 1.137 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2.767 & 0 \\ 0 & 2.767 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1.858 & 0 \\ 0 & 1.858 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 4.932 & 0 & 0.0261 & 0 \\ 0 & 4.932 & 0 & 0.0261 \\ 0.0261 & 0 & 2.726 & 0 \\ 0 & 0.0261 & 0 & 2.726 \end{bmatrix}, \quad G_1 = 0_{2 \times 2}, \quad G_2 = 0_{4 \times 4}.$$

On the other hand, for  $\epsilon = -0.001$  no solution to the inequalities (129) and (130) exists. Theorem 4.3 thus gives the same results as the direct stability analysis.

**Example 4.8** Let  $A(\rho) = A_0 + \rho A_1$  where

$$A_0 = \begin{bmatrix} 1.1132 & 1.6802 & -1.8252 & -0.5279 \\ 1.2328 & -0.8224 & -0.3503 & -0.8995 \\ 2.8858 & 1.9407 & -3.1417 & -1.1186 \\ 1.5929 & 0.1522 & -0.4807 & -2.0469 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -7.7372 & 0 & 0 \\ 7.7372 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the method of [94], one can show that the matrix  $A(\rho)$  is Hurwitz if and only if  $\rho \in (-0.9688, 0.5024)$ . In this example,  $n = 4$ ,  $r = \text{rank}(A_1) = 2$  and  $m = \frac{1}{2}(2nr - r^2 + r) = 7$ . The parameter-dependent Lyapunov matrix  $P(\rho) = \sum_{i=0}^7 \rho^i P_i$  where the matrices

$P_i, i = 0, 1, \dots, 7$  are

$$\begin{aligned}
P_0 &= \begin{bmatrix} 0.2259 & 0.1151 & 0.2201 & 0.1185 \\ 0.1151 & 0.0782 & 0.1253 & 0.0589 \\ 0.2201 & 0.1253 & 0.2430 & 0.1199 \\ 0.1185 & 0.0589 & 0.1199 & 0.0779 \end{bmatrix}, & P_1 &= \begin{bmatrix} -1.3999 & 0.5126 & -0.7016 & -0.7606 \\ 0.5126 & 0.9097 & 1.0028 & 0.3668 \\ -0.7016 & 1.0028 & 0.0753 & -0.3657 \\ -0.7606 & 0.3668 & -0.3657 & -0.5246 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 25.463 & 6.909 & 19.245 & 11.388 \\ 6.909 & 14.698 & 12.340 & 1.360 \\ 19.245 & 12.340 & 23.776 & 8.885 \\ 11.388 & 1.360 & 8.885 & 9.393 \end{bmatrix}, & P_3 &= \begin{bmatrix} -50.590 & 16.443 & -47.871 & -5.090 \\ 16.443 & 25.857 & 39.732 & 34.048 \\ -47.871 & 39.732 & -21.773 & -0.493 \\ -5.090 & 34.048 & -0.493 & -5.181 \end{bmatrix}, \\
P_4 &= \begin{bmatrix} -505.90 & 164.43 & -478.71 & -50.90 \\ 164.43 & 258.57 & 397.32 & 340.48 \\ -478.71 & 397.32 & -217.73 & -4.93 \\ -50.90 & 340.48 & -4.93 & -51.81 \end{bmatrix}, & P_5 &= \begin{bmatrix} -302.77 & 131.28 & -263.21 & -28.91 \\ 131.28 & 92.38 & 403.28 & 216.23 \\ -263.21 & 403.28 & -15.28 & 5.72 \\ -28.91 & 216.23 & 5.72 & -39.35 \end{bmatrix}, \\
P_6 &= \begin{bmatrix} 1049.6 & 0 & 0 & 0 \\ 0 & 1049.6 & 0 & 0 \\ 0 & 0 & -56.6 & 22.5 \\ 0 & 0 & 22.5 & -79.8 \end{bmatrix}, & P_7 &= 0_{4 \times 4}
\end{aligned} \tag{133}$$

satisfies the matrix inequality  $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$  for all  $\rho \in \mathbb{R}$  but it is positive-definite only when  $\rho \in (-0.9688, 0.5024)$ . On the other hand, the matrix inequalities (129) and (130), have no solution. This is expected, since  $[-1, 1]$  is not a subset of  $(-0.9688, 0.5024)$ .

Let now  $A(\rho) = A_0 + \rho A'_1$  and  $A'_1 = 0.5A_1$ . The exact stability domain for this system is  $(-1.9376, 1.0048)$ . Applying the algorithm of Theorem 4.3, and using the MATLAB<sup>TM</sup> LMI Toolbox [28], it can be verified that the inequalities (129) and (130) are indeed feasible. This result agrees with the direct analysis, since  $[-1, 1] \subset (-1.9376, 1.0048)$  and thus  $A_0 + \rho A'_1$  is Hurwitz for all  $\rho \in [-1, 1]$ .

**Remark 4.5** Notice that when  $A(\rho)$  is nominally stable, i.e., when the matrix  $A_0$  is Hurwitz, the inequality (129) is not necessary. This is due to the fact that  $A_0$  Hurwitz along with inequality (130) guarantees that  $P(0) > 0$ . Also, (130) ensures that  $P(\rho) > 0$  for all  $\rho \in [-1, 1]$ ; see [41]. Assuming therefore nominal stability, one can discard the inequality (129), thus reducing considerably the number of variables in the convex feasibility problem of Theorem 4.3.

## 4.4 *Conclusions*

This chapter propose a class of parameter-dependent Lyapunov matrices  $P(\rho)$  which can be used to test the stability of linear, time-invariant, parameter-dependent (LTIPD) systems of the form  $\dot{x} = (A_0 + \rho A_1)x$  where  $\rho \in \Omega$ . The proposed Lyapunov matrix has polynomial dependence on the parameter  $\rho$  of a known degree and can be used to derive exact (i.e., necessary and sufficient) conditions for the stability of LTIPD systems. We show that checking these conditions over a compact interval can be cast as a convex programming problem in terms of linear matrix inequalities without conservatism. Finally, it should be pointed out that the results of [55] as well as of the Example 4.8 indicate that the degree of the polynomial dependence given in Theorem 4.2 is only an upper bound (not tight) and the question of the lowest degree polynomial Lyapunov matrix is still open.

## CHAPTER V

### A STABILITY ANALYSIS TEST FOR ONE-PARAMETER POLYNOMIALLY-DEPENDENT LINEAR SYSTEMS

#### 5.1 *Introduction*

In this chapter, we develop a new approach for testing the stability of single-parameter polynomially-dependent LTI systems in the form

$$\dot{x} = A(\rho)x, \quad A(\rho) = \sum_{i=0}^N \rho^i A_i, \quad \rho \in [-1, +1], \quad A_i \in \mathbb{R}^{n \times n}, \quad i = 0, 1, 2, \dots, N. \quad (134)$$

For single-parameter affine-dependent LTI systems,

$$\dot{x} = (A_0 + \rho A_1)x, \quad \rho \in [-1, +1], \quad A_0, A_1 \in \mathbb{R}^{n \times n}, \quad (135)$$

Reference [24] gave the maximal stability interval around the origin when  $A_0$  is Hurwitz. Chapter 3 has already presented an approach to calculate the whole stability domain of system (135), which may be an open interval or a union of a finite number of disjointed open intervals of  $\mathbb{R}$ , even for the case when  $A_0$  is not Hurwitz. References [7, 64, 65] defined the concept of the guardian map and used it to give necessary and sufficient stability conditions for systems (134) and (135). The extension of such stability conditions to the synthesis problem is not directly evident. For affine, single-parameter LTIPD systems as in (135), in Chapter 4 we have already given a nonconservative and computable stability condition, which is amenable to synthesis. It is desired to generalize such LMI stability conditions for the one-parameter polynomially-dependent LTIPD systems as in (134).

The main contribution of this chapter is that it suggests a new approach for constructing an augmented affine LTIPD system in the form of (135) whose stability is equivalent to that of the original polynomially-dependent LTIPD system in (134). Thus, the method of

Chapter 4 can be used to give a necessary and sufficient, easily computable (in terms of LMIs) stability condition for the system in (134).

## 5.2 Equivalent Affine LTIPD Systems for Stability Analysis

Consider the matrix  $A(\rho)$  of polynomial degree  $N$  in (134), given as follows

$$A(\rho) = A_0 + \rho A_1 + \rho^2 A_2 + \cdots + \rho^N A_N \quad (136)$$

Let  $\deg A(\rho)$  denote the degree of  $A$  in terms of  $\rho$ . Clearly  $\deg A = N$  for the matrix valued function  $A(\rho)$  in (136). Decompose the matrix  $A(\rho)$  in its even and odd polynomial parts as follows

$$A(\rho) = A_0^a(\rho^2) + \rho A_1^a(\rho^2) \quad (137)$$

where  $A_0^a(\cdot)$  and  $A_1^a(\cdot)$  are polynomial matrices in terms of  $\rho^2$ . For example, if  $N$  is odd, then

$$A_0^a(\rho^2) := A_0 + \rho^2 A_2 + \rho^4 A_4 \cdots + \rho^{N-1} A_{N-1} \quad (138a)$$

$$A_1^a(\rho^2) := A_1 + \rho^2 A_3 + \rho^4 A_5 \cdots + \rho^{N-1} A_N \quad (138b)$$

whereas if  $N$  is even,

$$A_0^a(\rho^2) := A_0 + \rho^2 A_2 + \rho^4 A_4 + \cdots + \rho^N A_N \quad (139a)$$

$$A_1^a(\rho^2) := A_1 + \rho^2 A_3 + \rho^4 A_5 + \cdots + \rho^{N-2} A_{N-1} \quad (139b)$$

**Lemma 5.1** *Let the matrix  $A(\rho)$  in (136) with  $A_0$  Hurwitz. Then the matrix  $A(\rho)$  is Hurwitz for all  $\rho \in [-1, +1]$  if and only if the matrix*

$$\begin{pmatrix} A_0^a(r) & r A_1^a(r) \\ A_1^a(r) & A_0^a(r) \end{pmatrix} \quad (140)$$

*is Hurwitz for all  $r \in [0, 1]$ .*

*Proof.* Let  $\bar{\mathbb{C}}^+$  denote the closed right-half of the complex plane. The matrix  $A(\rho)$  is Hurwitz for all  $\rho \in [-1, +1]$  if and only if  $\det(sI - A_0^a(\rho^2) - \rho A_1^a(\rho^2)) \neq 0$  for all  $\rho \in [-1, +1]$  and for all  $s \in \bar{\mathbb{C}}^+$  which holds if and only if  $\det(sI - A_0^a(\rho^2) \pm \rho A_1^a(\rho^2)) \neq 0$  for all  $\rho \in [0, 1]$  and for all  $s \in \bar{\mathbb{C}}^+$ .

From the identity

$$\begin{aligned} & \begin{pmatrix} I & -\rho I \\ 0 & I \end{pmatrix} \begin{pmatrix} sI - A_0^a(\rho^2) & -\rho^2 A_1^a(\rho^2) \\ -A_1^a(\rho^2) & sI - A_0^a(\rho^2) \end{pmatrix} \begin{pmatrix} I & \rho I \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} sI - A_0^a(\rho^2) + \rho A_1^a(\rho^2) & 0 \\ -A_1^a(\rho^2) & sI - A_0^a(\rho^2) - \rho A_1^a(\rho^2) \end{pmatrix} \end{aligned}$$

it follows that  $\det(sI - A_0^a(\rho^2) \pm \rho A_1^a(\rho^2)) \neq 0$  for all  $\rho \in [0, 1]$  and for all  $s \in \bar{\mathbb{C}}^+$  if and only if

$$\det \begin{pmatrix} sI - A_0^a(\rho^2) & -\rho^2 A_1^a(\rho^2) \\ -A_1^a(\rho^2) & sI - A_0^a(\rho^2) \end{pmatrix} \neq 0, \quad \forall \rho \in [0, 1], \quad \forall s \in \bar{\mathbb{C}}^+ \quad (141)$$

The last condition is equivalent to the statement that the matrix in (140) is Hurwitz for all  $r \in [0, 1]$ . ■

**Theorem 5.1** *The matrix  $A(\rho)$  in (136) is Hurwitz for all  $\rho \in [-1, +1]$  if and only if the matrix*

$$A^{(1)}(\rho) := \begin{pmatrix} A_0^a\left(\frac{\rho+1}{2}\right) & \frac{\rho+1}{2} A_1^a\left(\frac{\rho+1}{2}\right) \\ A_1^a\left(\frac{\rho+1}{2}\right) & A_0^a\left(\frac{\rho+1}{2}\right) \end{pmatrix} \quad (142)$$

*is Hurwitz for all  $\rho \in [-1, +1]$ .*

*Proof.* It follows directly from Lemma 5.1 by setting  $r = (\rho + 1)/2$ . Then  $\rho \in [-1, +1]$  if and only if  $r \in [0, 1]$ . ■

Notice that the matrix in (142) depends polynomially on the parameter  $\rho$ . Hence the same procedure can be repeated for this matrix as well. Notice from (138) and (139) that  $\deg A_0^a \leq (N-1)/2$  and  $\deg A_1^a \leq (N-1)/2$  if  $N$  is odd, and  $\deg A_0^a \leq N/2$  and  $\deg A_1^a \leq (N-2)/2$



if  $N$  is even. It follows from (142) that  $\deg A^{(1)} \leq \max\{\deg A_0^a, \deg A_1^a + 1\} = \lfloor (N+1)/2 \rfloor$ . Therefore, the polynomial dependence of the new matrix  $A^{(1)}(\rho)$  has been reduced by a factor of two.

Specifically, one can rewrite the matrix in (142) as

$$A^{(1)}(\rho) = A_0^{(1)} + \rho A_1^{(1)} + \rho^2 A_2^{(1)} + \cdots + \rho^{N_1} A_{N_1}^{(1)} \quad (143)$$

where  $N_1 = \lfloor (N+1)/2 \rfloor$  for some matrices  $A_j^{(1)} \in \mathbb{R}^{2n \times 2n}$  and  $j = 0, 1, 2, \dots, N_1$ . This procedure will lead after *at most*  $q_{\max} := \lfloor \log_2 N \rfloor + 1$  steps to an *affine* in the parameter  $\rho$  system

$$A^{(q_{\max})}(\rho) = A_0^{(q_{\max})} + \rho A_1^{(q_{\max})} \quad (144)$$

for some constant matrices  $A_0^{(q_{\max})}, A_1^{(q_{\max})} \in \mathbb{R}^{2^{q_{\max}} n \times 2^{q_{\max}} n}$ .

The following is thus immediate from the previous iterative procedure.

**Corollary 5.1** *The matrix  $A(\rho)$  in (136), with  $A_0$  Hurwitz, is Hurwitz for all  $\rho \in [-1, +1]$  if and only if the matrices  $A^{(q)}(\rho)$  are Hurwitz for all  $\rho \in [-1, +1]$  and all  $q = 1, 2, \dots, q_{\max}$ .*

This corollary allows one to check the stability of the polynomial matrix  $A(\rho)$  for all  $\rho \in [-1, +1]$  by checking the stability of the *affine* matrix  $A^{(q_{\max})}(\rho)$  for all  $\rho \in [-1, +1]$  instead.

### 5.3 LMI Stability Condition for Parameter-Dependent LTI Systems

With Corollary 5.1, the stability of  $A(\rho)$  in (136) is equivalent to the stability of  $A^{(q_{\max})}(\rho)$  in (144) over the compact set  $[-1, +1]$ . Thus, one can use the LMI stability condition in Corollary 4.3 of Chapter 4 to test the stability of  $A^{(q_{\max})}(\rho)$ , and therefore the stability of  $A(\rho)$  in (136) over the compact set  $[-1, +1]$ .

## 5.4 Numerical Examples

**Example 5.1** Consider the matrix-valued function  $A(\rho)$  as follows, which depends polynomially on the scalar  $\rho$ .

$$A(\rho) = \begin{bmatrix} \rho^2 + 1 - 2(\rho + 1)^4 & -\rho^2 - 1 + (\rho + 1)^4 \\ 2\rho^2 + 2 - 2(\rho + 1)^4 & -2\rho^2 - 2 + (\rho + 1)^4 \end{bmatrix} \quad (145)$$

A direct eigenvalue analysis gives

$$\lambda_1(A(\rho)) = -1 - \rho^2, \quad \lambda_2(A(\rho)) = -(1 + \rho)^4.$$

Therefore, the matrix  $A(\rho)$  is Hurwitz for  $\rho \in (-1, +1]$  and is not Hurwitz for  $\rho = -1$ .

Applying Corollary 5.1 to  $A(\rho)$  in (145), one has

$$\begin{aligned} A(\rho) &= A_0 + \rho A_1 + \rho^2 A_2 + \rho^3 A_3 + \rho^4 A_4 = A_0^a(\rho^2) + \rho A_1^a(\rho^2) \\ A_0^a(\rho^2) &= A_0 + \rho^2 A_2 + \rho^4 A_4, \quad A_1^a(\rho^2) = A_1 + \rho^2 A_3 \end{aligned}$$

and

$$\begin{aligned} A^{(1)}(\rho) &= \begin{bmatrix} A_0 + \left(\frac{\rho+1}{2}\right)A_2 + \left(\frac{\rho+1}{2}\right)^2 A_4 & \left(\frac{\rho+1}{2}\right)A_1 + \left(\frac{\rho+1}{2}\right)^2 A_3 \\ A_1 + \left(\frac{\rho+1}{2}\right)A_3 & A_0 + \left(\frac{\rho+1}{2}\right)A_2 + \left(\frac{\rho+1}{2}\right)^2 A_4 \end{bmatrix} \\ &= A_0^{(1)} + \rho A_1^{(1)} + \rho^2 A_2^{(1)}, \end{aligned}$$

furthermore

$$\begin{aligned} A^{(2)}(\rho) &= \begin{bmatrix} A_0^{(1)} + \left(\frac{\rho+1}{2}\right)A_2^{(1)} & \left(\frac{\rho+1}{2}\right)A_1^{(1)} \\ A_1^{(1)} & A_0^{(1)} + \left(\frac{\rho+1}{2}\right)A_2^{(1)} \end{bmatrix} \\ &= A_0^{(2)} + \rho A_1^{(2)}. \end{aligned}$$

The numerical values of  $A_0^{(1)}$ ,  $A_1^{(1)}$ ,  $A_2^{(1)}$ ,  $A_0^{(2)}$ ,  $A_1^{(2)}$  are given as

$$A_0^{(1)} = \begin{bmatrix} -7 & 2.75 & -6 & 3 \\ -5.5 & 1.25 & -6 & 3 \\ -12 & 6 & -7 & 2.75 \\ -12 & 6 & -5.5 & 1.25 \end{bmatrix}, \quad A_1^{(1)} = \begin{bmatrix} -6.5 & 3 & -8 & 4 \\ -6 & 2.5 & -8 & 4 \\ -4 & 2 & -6.5 & 3 \\ -4 & 2 & -6 & 2.5 \end{bmatrix}, \quad A_2^{(1)} = \begin{bmatrix} -0.5 & 0.25 & -2 & 1 \\ -0.5 & 0.25 & -2 & 1 \\ 0 & 0 & -0.5 & 0.25 \\ 0 & 0 & -0.5 & 0.25 \end{bmatrix}$$

$$A_0^{(2)} = \begin{bmatrix} -7.250 & 2.875 & -7.000 & 3.500 & -3.250 & 1.500 & -4.000 & 2.000 \\ -5.750 & 1.375 & -7.000 & 3.500 & -3.000 & 1.250 & -4.000 & 2.000 \\ -12.000 & 6.000 & -7.250 & 2.875 & -2.000 & 1.000 & -3.250 & 1.500 \\ -12.000 & 6.000 & -5.750 & 1.375 & -2.000 & 1.000 & -3.000 & 1.250 \\ -6.500 & 3.000 & -8.000 & 4.000 & -7.250 & 2.875 & -7.000 & 3.500 \\ -6.000 & 2.500 & -8.000 & 4.000 & -5.750 & 1.375 & -7.000 & 3.500 \\ -4.000 & 2.000 & -6.500 & 3.000 & -12.000 & 6.000 & -7.250 & 2.875 \\ -4.000 & 2.000 & -6.000 & 2.500 & -12.000 & 6.000 & -5.750 & 1.375 \end{bmatrix}$$

$$A_1^{(2)} = \begin{bmatrix} -0.250 & 0.125 & -1.000 & 0.500 & -3.250 & 1.500 & -4.000 & 2.000 \\ -0.250 & 0.125 & -1.000 & 0.500 & -3.000 & 1.250 & -4.000 & 2.000 \\ 0 & 0 & -0.250 & 0.125 & -2.000 & 1.000 & -3.250 & 1.500 \\ 0 & 0 & -0.250 & 0.125 & -2.000 & 1.000 & -3.000 & 1.250 \\ 0 & 0 & 0 & 0 & -0.250 & 0.125 & -1.000 & 0.500 \\ 0 & 0 & 0 & 0 & -0.250 & 0.125 & -1.000 & 0.500 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.250 & 0.125 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.250 & 0.125 \end{bmatrix}$$

Applying Corollary 5.1 to  $A(\rho)$  in (145) shows that  $A(\rho)$  is Hurwitz for all  $\rho \in [-1, +1]$  if and only if  $A^{(2)}(\rho) = A_0^{(2)} + \rho A_1^{(2)}$  is Hurwitz for all  $\rho \in [-1, +1]$ . The stability of  $A^{(2)}(\rho)$  over the compact set  $[-1, +1]$  can be checked by either of the following two methods.

(a) **Whole Stability Domain**

Using the approach of Chapter 3, the whole stability domain for  $A^{(2)}(\rho) = A_0^{(2)} + \rho A_1^{(2)}$  can be computed as

$$\Omega = (-32.891477, -4.907828) \bigcup (-1.226272, 0.99198) \bigcup (1.017235, 2.608081)$$

Thus,  $A^{(2)}(\rho)$  is not Hurwitz for all  $\rho \in [-1, +1]$  since  $[-1, +1]$  is not a subset of  $\Omega$ , and therefore  $A(\rho)$  is not Hurwitz for all  $\rho \in [-1, +1]$ .

(b) **LMI Stability Condition**

Applying Corollary 4.3 of Chapter 4 to the case  $A^{(2)}(\rho) = A_0^{(2)} + \rho A_1^{(2)}$  with  $n = 8$ ,  $m = \frac{1}{2}n(n+1) - 1 = 35$ ,  $k = \lceil \frac{m}{2} \rceil + 1 = 19$ , one cannot find symmetric matrices  $P_\Sigma \in \mathbb{R}^{nk \times nk}$ ,  $D_1 \in \mathbb{R}^{n(k-1) \times n(k-1)}$  and  $D_2 \in \mathbb{R}^{nk \times nk}$  and skew-symmetric matrices

$G_1 \in \mathbb{R}^{n(k-1) \times n(k-1)}$ ,  $G_2 \in \mathbb{R}^{nk \times nk}$ , such that LMI (129) and LMI (130) are satisfied. Thus,  $A^{(2)}(\rho)$  is not Hurwitz for all  $\rho \in [-1, +1]$  and furthermore  $A(\rho)$  is not Hurwitz for all  $\rho \in [-1, +1]$ .

**Example 5.2** Consider the polynomially-dependent matrix  $A(\rho)$  as follows

$$A(\rho) = \begin{bmatrix} 1 - \rho^2 - 2(\rho + 1)^4 & -1 + \rho^2 + (\rho + 1)^4 \\ 2 - 2\rho^2 - 2(\rho + 1)^4 & -2 + 2\rho^2 + (\rho + 1)^4 \end{bmatrix} \quad (146)$$

A direct eigenvalue analysis yields

$$\lambda_1(A(\rho)) = -1 + \rho^2, \quad \lambda_2(A(\rho)) = -(1 + \rho)^4.$$

Therefore  $A(\rho)$  is Hurwitz for  $\rho \in (-1, +1)$  and it is not Hurwitz for  $\rho = \pm 1$ .

Applying Corollary 5.1 to  $A(\rho)$  in (146), and following the same procedure as in Example 5.1, one obtains

$$A^{(1)}(\rho) = A_0^{(1)} + \rho A_1^{(1)} + \rho^2 A_2^{(1)}, \quad A^{(2)}(\rho) = A_0^{(2)} + \rho A_1^{(2)}.$$

The numerical values of these matrices are given below.

$$A_0^{(1)} = \begin{bmatrix} -8 & 3.75 & -6 & 3 \\ -7.5 & 3.25 & -6 & 3 \\ -12 & 6 & -8 & 3.75 \\ -12 & 6 & -7.5 & 3.25 \end{bmatrix}, \quad A_1^{(1)} = \begin{bmatrix} -7.5 & 4 & -8 & 4 \\ -8 & 4.5 & -8 & 4 \\ -4 & 2 & -7.5 & 4 \\ -4 & 2 & -8 & 4.5 \end{bmatrix}, \quad A_2^{(1)} = \begin{bmatrix} -0.5 & 0.25 & -2 & 1 \\ -0.5 & 0.25 & -2 & 1 \\ 0 & 0 & -0.5 & 0.25 \\ 0 & 0 & -0.5 & 0.25 \end{bmatrix}$$

$$A_0^{(2)} = \begin{bmatrix} -8.250 & 3.875 & -7.000 & 3.500 & -3.750 & 2.000 & -4.000 & 2.000 \\ -7.750 & 3.375 & -7.000 & 3.500 & -4.000 & 2.250 & -4.000 & 2.000 \\ -12.000 & 6.000 & -8.250 & 3.875 & -2.000 & 1.000 & -3.750 & 2.000 \\ -12.000 & 6.000 & -7.750 & 3.375 & -2.000 & 1.000 & -4.000 & 2.250 \\ -7.500 & 4.000 & -8.000 & 4.000 & -8.250 & 3.875 & -7.000 & 3.500 \\ -8.000 & 4.500 & -8.000 & 4.000 & -7.750 & 3.375 & -7.000 & 3.500 \\ -4.000 & 2.000 & -7.500 & 4.000 & -12.000 & 6.000 & -8.250 & 3.875 \\ -4.000 & 2.000 & -8.000 & 4.500 & -12.000 & 6.000 & -7.750 & 3.375 \end{bmatrix}$$

$$A_1^{(2)} = \begin{bmatrix} -0.250 & 0.125 & -1.000 & 0.500 & -3.750 & 2.000 & -4.000 & 2.000 \\ -0.250 & 0.125 & -1.000 & 0.500 & -4.000 & 2.250 & -4.000 & 2.000 \\ 0 & 0 & -0.250 & 0.125 & -2.000 & 1.000 & -3.750 & 2.000 \\ 0 & 0 & -0.250 & 0.125 & -2.000 & 1.000 & -4.000 & 2.250 \\ 0 & 0 & 0 & 0 & -0.250 & 0.125 & -1.000 & 0.500 \\ 0 & 0 & 0 & 0 & -0.250 & 0.125 & -1.000 & 0.500 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.250 & 0.125 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.250 & 0.125 \end{bmatrix}$$

The stability of  $A^{(2)}(\rho)$  and  $A(\rho)$  over the compact set  $[-1, +1]$  can be checked by either of the following two methods.

(a) **Whole Stability Domain**

Using the approach of Chapter 3, the whole stability domain for  $A^{(2)}(\rho) = A_0^{(2)} + \rho A_1^{(2)}$  is computed as follows

$$\Omega = (-32.891477, -4.907828) \cup (-1.226272, 0.9937)$$

It follows that  $A^{(2)}(\rho)$  is not Hurwitz for all  $\rho \in [-1, +1]$  since  $[-1, +1]$  is not a subset of  $\Omega$ . Therefore  $A(\rho)$  is not Hurwitz for all  $\rho \in [-1, +1]$ .

(b) **LMI Stability Condition**

Applying Corollary 4.3 in Chapter 4 to the case  $A^{(2)}(\rho) = A_0^{(2)} + \rho A_1^{(2)}$  with  $n = 8$ ,  $m = \frac{1}{2}n(n+1) - 1 = 35$ ,  $k = \lceil \frac{m}{2} \rceil + 1 = 19$ , one cannot find symmetric matrices  $P_\Sigma \in \mathbb{R}^{nk \times nk}$ ,  $D_1 \in \mathbb{R}^{n(k-1) \times n(k-1)}$  and  $D_2 \in \mathbb{R}^{nk \times nk}$  and skew-symmetric matrices  $G_1 \in \mathbb{R}^{n(k-1) \times n(k-1)}$ ,  $G_2 \in \mathbb{R}^{nk \times nk}$ , such that LMI (129) and LMI (130) are satisfied. Therefore  $A^{(2)}(\rho)$  is not Hurwitz for all  $\rho \in [-1, +1]$  and thus  $A(\rho)$  is not Hurwitz for all  $\rho \in [-1, +1]$ .

**Example 5.3** Consider the polynomially-dependent matrix  $A(\rho)$  as follows

$$A(\rho) = \begin{bmatrix} -3 - \rho & -3 - \rho + (\rho + 2)^2 \\ 1 + \rho + \rho^2 & 1 + \rho + \rho^2 - (\rho + 2)^2 \end{bmatrix} \quad (147)$$

A direct eigenvalue analysis yields

$$\lambda_1(A(\rho)) = -2 + \rho^2, \quad \lambda_2(A(\rho)) = -(2 + \rho)^2.$$

Therefore  $A(\rho)$  is Hurwitz for  $\rho \in [-1, +1]$ .

Applying Corollary 5.1 to  $A(\rho)$  in (147), and following the same procedure as in Example 5.1, one obtains

$$A^{(1)}(\rho) = A_0^{(1)} + \rho A_1^{(1)}.$$

The numerical values of these matrices are given below.

$$A_0^{(1)} = \begin{bmatrix} -3 & 1.5 & -0.5 & 1.5 \\ 1.5 & -3 & 0.5 & -1.5 \\ -1 & 3 & -3 & 1.5 \\ 1 & -3 & 1.5 & -3 \end{bmatrix}, \quad A_1^{(1)} = \begin{bmatrix} 0 & 0.5 & -0.5 & 1.5 \\ 0.5 & 0 & 0.5 & -1.5 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 \end{bmatrix}$$

The stability of  $A^{(1)}(\rho)$  and  $A(\rho)$  over the compact set  $[-1, +1]$  can be checked by either of the following two methods.

(a) **Whole Stability Domain**

Using the approach of Chapter 3, the whole stability domain for  $A^{(1)}(\rho) = A_0^{(1)} + \rho A_1^{(1)}$  is computed as follows

$$\Omega = (-9, 3)$$

It follows that  $A^{(1)}(\rho)$  is Hurwitz for all  $\rho \in [-1, +1]$  since  $[-1, +1]$  is a subset of  $\Omega$ . Therefore  $A(\rho)$  is Hurwitz for all  $\rho \in [-1, +1]$ . It should be reminded that the stability domain of  $A^{(1)}(\rho)$  in most cases is different from that of  $A(\rho)$ .

(b) **LMI Stability Condition**

Applying Corollary 4.3 in Chapter 4 to the case  $A^{(1)}(\rho) = A_0^{(1)} + \rho A_1^{(1)}$  with  $n = 4$ ,  $m = \frac{1}{2}n(n+1) - 1 = 9$ ,  $k = \lceil \frac{m}{2} \rceil + 1 = 6$ , one can find symmetric matrices  $P_\Sigma \in \mathbb{R}^{nk \times nk}$ ,  $D_1 \in \mathbb{R}^{n(k-1) \times n(k-1)}$  and  $D_2 \in \mathbb{R}^{nk \times nk}$  i.e.,  $P_\Sigma \in \mathbb{R}^{24 \times 24}$ ,  $D_1 \in \mathbb{R}^{20 \times 20}$  and  $D_2 \in \mathbb{R}^{24 \times 24}$ , and skew-symmetric matrices  $G_1 \in \mathbb{R}^{n(k-1) \times n(k-1)}$ ,  $G_2 \in \mathbb{R}^{nk \times nk}$ ,

i.e.,  $G_1 \in \mathbb{R}^{20 \times 20}$ ,  $G_2 \in \mathbb{R}^{24 \times 24}$ , such that LMI (129) and LMI (130) are satisfied. The values of these matrices are not given here for the sake of brevity. Therefore  $A^{(1)}(\rho)$  is Hurwitz for all  $\rho \in [-1, +1]$  and thus  $A(\rho)$  is Hurwitz for all  $\rho \in [-1, +1]$ .

## 5.5 *Conclusions*

This chapter suggests an approach to test the stability of polynomially dependent LTIPD systems with a single parameter. The approach consists of constructing an augmented affine LTIPD system from the single-parameter polynomially-dependent LTIPD system. Stability of the new affine system over a given, compact set is proven to be equivalent to that of the original polynomially-dependent LTIPD system. Thus, the existing LMI stability tests of Chapter 4 for single-parameter affine LTIPD systems can be used to establish the stability of single-parameter polynomially-dependent LTIPD systems.

## CHAPTER VI

### MULTI-PARAMETER DEPENDENT LYAPUNOV FUNCTIONS FOR STABILITY ANALYSIS OF MULTI-PARAMETER LTIPD SYSTEMS

The objective of this chapter is to find computable, non-conservative conditions for checking the asymptotic stability of multi-parameter LTIPD systems of the form

$$\dot{x} = A(\rho)x, \quad A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2 + \dots + \rho_m A_m \quad (148)$$

where  $A_i \in \mathbb{R}^{n \times n}$ ,  $i = 0, 1, 2, \dots, m$  and  $\rho = [\rho_1, \rho_2, \dots, \rho_m]^T \in [-1, +1]^m$ .

Stability criteria for LTIPD systems can be derived through searching for a Lyapunov function, which can be constant or parameter-dependent. Most of current stability criteria are sufficient but not necessary [37, 12, 35, 40, 44, 56, 2, 73]. Using the concept of a power transformation of a square matrix [19], Zelentsovsky [88] developed sufficient conditions for the existence of the homogeneous polynomial Lyapunov function of a given degree for a linear system with box-bounded, time-varying, multi-parameter dependent uncertainty. For single-parameter LTIPD systems with the parameter inside a compact set, Chapter 4 suggested a nonconservative and computable LMI stability condition, using a polynomial-type Lyapunov function with an explicitly given bounded degree. It is desirable to generalize this result to the multi-parameter LTIPD system (148). Notice that the robust stability and performance analysis for multi-parameter LTIPD systems is, in general, an NP-hard problem [15, 62, 72, 87, 14].

This chapter offers two technical contributions. First, it is shown that the following multi-parameter dependent polynomial-type Lyapunov function of bounded, explicitly known degree can be used to derive nonconservative (i.e., necessary and sufficient) stability conditions



for the system in (148). The Lyapunov matrix is given by

$$P(\rho) = \sum_{i_1+\dots+i_m=0}^{i_1+\dots+i_m=K} \rho_1^{i_1} \cdots \rho_m^{i_m} P_{i_1,\dots,i_m}$$

where  $i_1, i_2, \dots, i_m$  are nonnegative integers. Second, the parameter-dependent matrices  $P(\rho)$  and  $R(\rho) = A^T(\rho)P(\rho) + P(\rho)A(\rho)$  can be rewritten as matrix-valued, polynomially parameter-dependent quadratic functions. The conditions for such matrix-valued quadratic functions to be positive definite or negative definite over a compact set are expressed in terms of LMIs without conservatism.

## 6.1 Preliminaries

Given a multi-parameter dependent polynomial  $p(\rho)$ , where  $\rho = [\rho_1, \rho_2, \dots, \rho_m]^T \in \mathbb{R}^m$ ,

$$p(\rho) = \sum \rho_1^{k_1} \cdots \rho_m^{k_m} p_{k_1,\dots,k_m},$$

we define the multi-parameter  $\rho$ -degree denoted by  $\deg(p(\rho))$  and the single-parameter  $\rho_i$ -degree denoted by  $\deg_{\rho_i}(p(\rho))$ ,  $i = 1, \dots, m$ , as follows,

$$\begin{aligned} \deg(p(\rho)) &= \max\{k_1 + \dots + k_m, \quad \text{s.t. } p_{k_1,\dots,k_m} \neq 0\}, \\ \deg_{\rho_i}(p(\rho)) &= \max\{k_i, \quad \text{s.t. } p_{\dots,k_i,\dots} \neq 0\} \end{aligned}$$

From these definitions, it is clear that  $\deg_{\rho_i}(p(\rho)) \leq \deg(p(\rho))$  for  $i = 1, 2, \dots, m$ .

**Example 6.1** Consider one multi-variable polynomial

$$p_1(\rho) = 2\rho_1^5 + 3\rho_1^3\rho_2^3 + 1.5\rho_2^4.$$

Then,

$$\deg_{\rho_1}(p_1(\rho)) = 5, \quad \deg_{\rho_2}(p_1(\rho)) = 4, \quad \deg(p_1(\rho)) = 6.$$

Consider the multi-variable polynomial

$$p_2(\rho) = 2.3\rho_1^5 + 3\rho_1^2\rho_2^3 + 1.5\rho_2^5.$$

Then,

$$\deg_{\rho_1}(p_2(\rho)) = \deg_{\rho_2}(p_2(\rho)) = \deg(p_2(\rho)) = 5.$$

The previous definitions can be generalized to the matrix-valued polynomials. Given a matrix-valued, multi-parameter dependent polynomial  $P(\rho) \in \mathbb{R}^{n \times n}$ , where  $\rho = [\rho_1, \rho_2, \dots, \rho_m]^T \in \mathbb{R}^m$ ,

$$P(\rho) = \sum \rho_1^{k_1} \cdots \rho_m^{k_m} P_{k_1, \dots, k_m},$$

the multi-parameter  $\rho$ -degree denoted by  $\deg(P(\rho))$  and the single-parameter  $\rho_i$ -degree denoted by  $\deg_{\rho_i}(P(\rho))$ ,  $i = 1, \dots, m$  are defined as follows,

$$\begin{aligned} \deg(P(\rho)) &= \max\{k_1 + \dots + k_m, \quad \text{s.t.} \quad P_{k_1, \dots, k_m} \neq 0\}, \\ \deg_{\rho_i}(P(\rho)) &= \max\{k_i, \quad \text{s.t.} \quad P_{\dots, k_i, \dots} \neq 0\} \end{aligned}$$

When we study  $\deg_{\rho_i}(p(\rho))$  or  $\deg_{\rho_i}(P(\rho))$ , all the parameters  $\rho_j$ ,  $j = 1, 2, \dots, m$ ,  $j \neq i$ , are treated as constants and Lemma 4.1 of Chapter 4 can be generalized to the multi-parameter case as follows.

**Lemma 6.1** *Given matrices  $A_i \in \mathbb{R}^{n \times n}$ ,  $i = 0, 1, 2, \dots, m$  and  $\rho = [\rho_1, \rho_2, \dots, \rho_m]^T \in \mathbb{R}^m$ , let  $p(\rho)$  denote the polynomial in  $\rho$  defined by  $p(\rho) := \det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)$ . Then  $\deg_{\rho_i}(p(\rho)) \leq n$  for  $i = 1, 2, \dots, m$ . Moreover, if  $\text{rank}(A_i) = r_i < n$  for  $i = 1, 2, \dots, m$ , then  $\deg_{\rho_i}(p(\rho)) \leq r_i$ .*

*Proof.* Let

$$\Sigma_0 := A_0 + \sum_{j=1, j \neq i}^m \rho_j A_j.$$

Since  $\rho_j$ ,  $j = 1, 2, \dots, m$ ,  $j \neq i$  and  $\Sigma_0$  are treated as constants, applying Lemma 4.1 to  $p(\rho) = \det(\Sigma_0 + \rho_i A_i)$ , one obtains  $\deg_{\rho_i}(p(\rho)) \leq n$  and that if  $\text{rank}(A_i) = r_i < n$ , then  $\deg_{\rho_i}(p(\rho)) \leq r_i$ . ■

**Example 6.2** Consider the two-parameter dependent matrix  $A(\rho_1, \rho_2) = A_0 + \rho_1 A_1 + \rho_2 A_2$ , where

$$A_0 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}.$$

Notice that

$$\text{rank}(A_1) = 1, \quad \text{rank}(A_2) = 2.$$

Since

$$p(\rho) = \det(A_0 + \rho_1 A_1 + \rho_2 A_2) = -5 + \rho_1 - 8\rho_2 - \rho_2^2$$

we get

$$\deg_{\rho_1}(p(\rho)) = 1, \quad \deg_{\rho_2}(p(\rho)) = 2.$$

The following lemma concerns the degree of the multi-parameter polynomial  $\det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)$ .

**Lemma 6.2** *Given matrices  $A_i \in \mathbb{R}^{n \times n}$ ,  $i = 0, 1, 2, \dots, m$  and  $\rho = [\rho_1, \rho_2, \dots, \rho_m]^T \in \mathbb{R}^m$ , let  $p(\rho)$  denote the polynomial in  $\rho$  defined by  $p(\rho) := \det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)$ . Then  $\deg(p(\rho)) \leq n$ . Moreover, if  $\dim(\mathcal{N}(A_1) \cap \mathcal{N}(A_2) \cap \dots \cap \mathcal{N}(A_m)) = r < n$ , then  $\deg(p(\rho)) \leq n - r$ , where  $\mathcal{N}(A_i)$  is the null space of matrix  $A_i$ ,  $i = 0, 1, 2, \dots, m$ .*

*Proof.* The determinant of a matrix  $F \in \mathbb{R}^{n \times n}$  can be computed from [69]

$$\det F = \sum_{a_1 \neq a_2 \neq \dots \neq a_n} \pm (F_{1,a_1} F_{2,a_2} \dots F_{n,a_n}). \quad (149)$$

The determinant of  $F$  is thus a sum of  $n!$  terms, each term being the product of  $n$  elements. Moreover, each of these elements is chosen from a different row and column of the matrix  $F$ . Let  $F = A_0 + \rho_1 A_1 + \dots + \rho_m A_m$ . Then

$$\det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m) = \sum_{a_1 \neq a_2 \neq \dots \neq a_n} \pm (F_{1,a_1} F_{2,a_2} \dots F_{n,a_n}), \quad (150)$$

where

$$F_{k,a_k} = A_{0,(k,a_k)} + \rho_1 A_{1,(k,a_k)} + \dots + \rho_m A_{m,(k,a_k)}, \quad k = 1, 2, \dots, n$$

where  $A_{i,(k,a_k)}$  denotes the  $(k, a_k)$  element of the matrix  $A_i$ . For every permutation  $(a_1, a_2, \dots, a_n)$ , we have

$$\deg(F_{1,a_1} F_{2,a_2} \dots F_{n,a_n}) \leq n.$$

Together with (150), one has

$$\deg(\det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)) \leq n$$

If  $\dim(\mathcal{N}(A_1) \cap \mathcal{N}(A_2) \cap \dots \cap \mathcal{N}(A_m)) = r < n$ , there exist  $r$  linearly independent vectors  $v_1, v_2, \dots, v_r \in \mathbb{R}^n$  such that

$$A_i v_j = 0, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, r$$

Choose an additional  $n - r$  linearly independent vectors  $u_1, u_2, \dots, u_{n-r} \in \mathbb{R}^n$  such that the matrix

$$T = \begin{bmatrix} u_1 & u_2 & \dots & u_{n-r} & v_1 & v_2 & \dots & v_r \end{bmatrix}$$

is nonsingular. Furthermore,

$$\begin{aligned} \det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m) &= \det(T^{-1}(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)T) \\ &= \det(T^{-1}) \det((A_0 + \rho_1 A_1 + \dots + \rho_m A_m)T) \\ &= \det(T^{-1}) \det\left( \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \dots & \bar{u}_{n-r} & \bar{v}_1 & \bar{v}_2 & \dots & \bar{v}_r \end{bmatrix} \right) \end{aligned}$$

where the column vectors  $\bar{u}_i$  and  $\bar{v}_i$  are given by

$$\begin{aligned} \bar{u}_i &= (A_0 + \rho_1 A_1 + \dots + \rho_m A_m) u_i \\ &= (A_0 u_i) + \rho_1 (A_1 u_i) + \dots + \rho_m (A_m u_i), \quad i = 1, 2, \dots, n - r \\ \bar{v}_i &= A_0 v_i, \quad i = 1, 2, \dots, r \end{aligned}$$

Since  $\bar{v}_i$  is constant, together with formula (149), one has

$$\begin{aligned} &\det\left( \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \dots & \bar{u}_{n-r} & \bar{v}_1 & \bar{v}_2 & \dots & \bar{v}_r \end{bmatrix} \right) \\ &= \sum_{a_1 \neq a_2 \neq \dots \neq a_n} \pm (\bar{u}_{1,a_1} \bar{u}_{2,a_2} \dots \bar{u}_{n-r,a_{n-r}} \bar{v}_{1,a_{n-r}+1} \dots \bar{v}_{r,a_n}) \end{aligned}$$

For every permutation  $(a_1, a_2, \dots, a_n) \in \{1, 2, \dots, n\}$ , we have that

$$\deg(\bar{u}_{1,a_1} \bar{u}_{2,a_2} \dots \bar{u}_{n-r,a_{n-r}} \bar{v}_{1,a_{n-r+1}} \dots \bar{v}_{r,a_n}) = \deg(\bar{u}_{1,a_1} \bar{u}_{2,a_2} \dots \bar{u}_{n-r,a_{n-r}}) \leq n - r,$$

thus,

$$\deg(\det(\begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \dots & \bar{u}_{n-r} & \bar{v}_1 & \bar{v}_2 & \dots & \bar{v}_r \end{bmatrix})) \leq n - r$$

and therefore,  $\deg(\det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)) \leq n - r$ . ■

**Example 6.3** Consider the two-parameter matrix  $A(\rho_1, \rho_2) = A_0 + \rho_1 A_1 + \rho_2 A_2$ , where

$$A_0 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 3 \\ 0 & -2 \end{bmatrix}.$$

It can be easily checked that

$$p(\rho) = \det(A_0 + \rho_1 A_1 + \rho_2 A_2) = -5 + \rho_1 - 8\rho_2$$

and  $\deg(p(\rho)) = 1$ . Moreover, the null space of the matrices  $A_1$  and  $A_2$  satisfy

$$\dim(\mathcal{N}(A_1) \cap \mathcal{N}(A_2)) = \dim(\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}) = 1$$

This example agrees with Lemma 6.2 since

$$\deg(\det(A_0 + \rho_1 A_1 + \rho_2 A_2)) \leq n - \dim(\mathcal{N}(A_1) \cap \mathcal{N}(A_2)).$$

**Lemma 6.3** Given matrices  $A_i \in \mathbb{R}^{n \times n}$ ,  $i = 0, 1, 2, \dots, m$  and  $\rho = [\rho_1, \rho_2, \dots, \rho_m]^T \in \mathbb{R}^m$ ,

if  $\text{rank}(A_i) = r_i$ ,  $i = 1, 2, \dots, m$  and  $\dim(\mathcal{N}(A_1) \cap \mathcal{N}(A_2) \cap \dots \cap \mathcal{N}(A_m)) = r < n$ , then

$$\text{Adj}(A_0 + \rho_1 A_1 + \dots + \rho_m A_m) = \sum_{k_1=k_2=\dots=k_m=0}^{k_1+k_2+\dots+k_m \leq \min\{n-1, n-r\}} \rho_1^{k_1} \rho_2^{k_2} \dots \rho_m^{k_m} N_{k_1, k_2, \dots, k_m}$$

that is,

$$\deg(\text{Adj}(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)) \leq \min\{n - 1, n - r\}$$

Moreover,

$$\deg_{\rho_i}(\text{Adj}(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)) \leq \min\{n - 1, r_i\}, \quad i = 1, 2, \dots, m$$

*Proof.* Recall from the definition of the adjoint of a matrix [38] that

$$[\text{Adj}(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)]_{ij} = (-1)^{i+j} \det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)_{[ji]}, \quad (151)$$

where  $(\cdot)_{[ji]}$  is the submatrix of  $(\cdot)$  in which the  $j$ -th row and the  $i$ -th column are eliminated and  $[\cdot]_{ij}$  is the  $ij$ -th element of the matrix  $[\cdot]$ . Since  $(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)_{[ji]} \in \mathbb{R}^{(n-1) \times (n-1)}$  and according to Lemma 6.2,

$$\deg\left(\det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)_{[ji]}\right) \leq n - 1 \quad (152)$$

for any possible pair  $[ji]$  and

$$\deg\left(\det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)\right) \leq n - r \quad (153)$$

Since

$$\deg\left(\det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)_{[ji]}\right) \leq \deg\left(\det(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)\right) \quad (154)$$

for any pair of  $[ji]$ , therefore, one has

$$\deg\left(\text{Adj}(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)\right) \leq \min\{n - 1, n - r\}. \quad (155)$$

While studying the single-parameter  $\rho_i$ -degree  $\deg_{\rho_i}(\cdot)$ , all the parameters  $\rho_j$ ,  $j = 1, 2, \dots, m$ ,  $j \neq i$  are treated as constants. Therefore, according to Lemma 4.2, one has

$$\deg_{\rho_i}\left(\text{Adj}(A_0 + \rho_1 A_1 + \dots + \rho_m A_m)\right) \leq \min\{n - 1, r_i\}, \quad i = 1, 2, \dots, m \quad (156)$$

■

**Example 6.4** Consider the two-parameter dependent matrix  $A(\rho_1, \rho_2) = A_0 + \rho_1 A_1 + \rho_2 A_2$ , where

$$A_0 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 3 \\ 0 & -2 \end{bmatrix}.$$

The null space of the matrices  $A_1$  and  $A_2$  satisfy

$$\dim(\mathcal{N}(A_1) \cap \mathcal{N}(A_2)) = \dim(\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}) = 1.$$

The adjoint of  $A(\rho_1, \rho_2)$  is computed as follows

$$\text{Adj}(A_0 + \rho_1 A_1 + \rho_2 A_2) = \begin{bmatrix} 1 + 3\rho_1 - 2\rho_2 & -3 - \rho_1 - 3\rho_2 \\ -2 & 1 \end{bmatrix}.$$

Let  $\text{Adj}(A_0 + \rho_1 A_1 + \rho_2 A_2)$  be denoted by  $P(\rho_1, \rho_2)$ . Then, it can be easily checked that

$$\deg(P(\rho_1, \rho_2)) = \deg_{\rho_1}(P(\rho_1, \rho_2)) = \deg_{\rho_2}(P(\rho_1, \rho_2)) = 1.$$

This result agrees with Lemma 6.3.

## 6.2 Multi-Parameter Dependent Lyapunov Functions

### 6.2.1 A Class of Multi-Parameter Dependent Lyapunov Functions

Theorem 6.1 in this section shows that a class of multi-parameter dependent Lyapunov functions can be used to study the stability of multi-parameter LTIPD systems (148) over a compact set in a nonconservative manner. Similarly to the proof of Theorem 4.1, Lemmas 2.1, 2.2 and 4.3 from Chapter 4, will be used in the proof of Theorem 6.1. In the following,  $\Omega$  denotes any subset of  $\mathbb{R}^m$ .

**Theorem 6.1** *Given matrices  $A_i \in \mathbb{R}^{n \times n}$ ,  $i = 0, 1, 2, \dots, m$ , and  $\rho = [\rho_1, \rho_2, \dots, \rho_m]^T \in \mathbb{R}^m$ , the following two statements are equivalent.*

(i)  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2 + \dots + \rho_m A_m$  is Hurwitz for all  $\rho \in \Omega \in \mathbb{R}^m$ .

(ii) There exist matrices  $P_{k_1, k_2, \dots, k_m} \in \mathbb{R}^{n \times n}$ , where  $k_i \in \mathbb{Z}_+^0$ ,  $i = 1, \dots, m$ , such that

$$A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0, \quad P(\rho) > 0, \quad \forall \rho \in \Omega \quad (157)$$

where

$$P(\rho) = \sigma(\rho) \sum_{k_1+k_2+\dots+k_m \leq n^2-1} \rho_1^{k_1} \rho_2^{k_2} \dots \rho_m^{k_m} P_{k_1, k_2, \dots, k_m}, \quad (158)$$

$$\text{i.e.,} \quad \deg(P(\rho)) \leq n^2 - 1 \quad \forall \rho \in \Omega$$

where  $\sigma(\rho) = -\text{sign}(\det(\bar{A}_0 + \rho_1 \bar{A}_1 + \dots + \rho_m \bar{A}_m))$

Furthermore, if  $\text{rank}(A_i) = r_i < n$ ,  $1 \leq i \leq m$ , then,

$$\deg_{\rho_i}(P(\rho)) \leq 2nr_i - r_i^2$$

If  $\dim(\mathcal{N}(A_1) \cap \mathcal{N}(A_2) \cap \dots \cap \mathcal{N}(A_m)) = r < n$ , then  $\deg(P(\rho)) \leq n^2 - r^2$ .

*Proof.* [Theorem 6.1] (ii)  $\Rightarrow$  (i). This is obvious.

(i)  $\Rightarrow$  (ii). First notice that  $\bar{A}(\rho) = \bar{A}_0 + \rho_1 \bar{A}_1 + \rho_2 \bar{A}_2 + \dots + \rho_m \bar{A}_m$ . Since  $A(\rho)$  is Hurwitz and according to Lemma 2.2,  $\det(\bar{A}_0 + \rho_1 \bar{A}_1 + \rho_2 \bar{A}_2 + \dots + \rho_m \bar{A}_m) \neq 0$ . Let  $|\cdot|$  denote the absolute value and choose the parameter-dependent, positive definite matrix, given by

$$Q(\rho) = |\det(\bar{A}_0 + \rho_1 \bar{A}_1 + \rho_2 \bar{A}_2 + \dots + \rho_m \bar{A}_m)| I_n = |\det(\bar{A}(\rho))| I_n > 0. \quad (159)$$

The inequality in (159) holds due to Lemma 2.2 and the fact that  $A(\rho)$  is Hurwitz for all  $\rho \in \Omega$ . Since  $A(\rho)$  is Hurwitz for all  $\rho \in \Omega$ , the following Lyapunov equation has a unique positive definite solution  $P(\rho) > 0$  for all  $\rho \in \Omega$  [98].

$$A(\rho)P(\rho) + P(\rho)A(\rho)^T + Q(\rho) = 0 \quad (160)$$

Solving this equation, one obtains

$$\bar{A}(\rho)\text{vec}(P) = -|\det(\bar{A}(\rho))|\text{vec}(I_n) \quad (161)$$

and thus

$$\begin{aligned} \text{vec}(P) &= -|\det(\bar{A}(\rho))|(\bar{A}(\rho))^{-1}\text{vec}(I_n) \\ &= -|\det(\bar{A}(\rho))|\frac{1}{\det(\bar{A}(\rho))}\text{Adj}(\bar{A}(\rho))\text{vec}(I_n) \\ &= \sigma(\rho)\text{Adj}(\bar{A}_0 + \rho_1 \bar{A}_1 + \rho_2 \bar{A}_2 + \dots + \rho_m \bar{A}_m)\text{vec}(I_n), \end{aligned} \quad (162)$$

where  $\sigma(\rho) = -\text{sign}(\det(\bar{A}(\rho)))$  and where the concept of  $\text{vec}(\cdot)$  is standard, see, for example, [98].  $\text{Adj}(\bar{A}_0 + \rho_1 \bar{A}_1 + \rho_2 \bar{A}_2 + \dots + \rho_m \bar{A}_m)$  is a polynomial in  $\rho$  and it follows from Lemma 6.3 that

$$\text{Adj}(\bar{A}_0 + \rho_1 \bar{A}_1 + \rho_2 \bar{A}_2 + \dots + \rho_m \bar{A}_m) = \sum_{k_1+k_2+\dots+k_m \leq n^2-1} \rho_1^{k_1} \rho_2^{k_2} \dots \rho_m^{k_m} N_{k_1, k_2, \dots, k_m} \quad (163)$$



where  $N_{k_1, k_2, \dots, k_m}$  are constant matrices. Together with (162), one has

$$P(\rho) = \sigma(\rho) \sum_{k_1=k_2=\dots=k_m=0}^{k_1+k_2+\dots+k_m \leq n^2-1} \rho_1^{k_1} \rho_2^{k_2} \dots \rho_m^{k_m} P_{k_1, k_2, \dots, k_m}$$

i.e.,

$$\deg(P(\rho)) \leq n^2 - 1,$$

where  $P_{k_1, k_2, \dots, k_m}$ ,  $0 \leq k_i \leq n^2 - 1$  are constant matrices and

$$P_{k_1, k_2, \dots, k_m} = \text{vec}^{-1}(N_{k_1, k_2, \dots, k_m} \text{vec}(I_n)), \quad k_i = 0, 1, 2, \dots, n^2 - 1 \quad (164)$$

where  $\text{vec}^{-1}(\cdot)$  is the inverse mapping of  $\text{vec}(\cdot)$  and matrices  $N_{k_1, k_2, \dots, k_m}$ ,  $k_i = 0, 1, 2, \dots, n^2 - 1$  can be calculated according to Corollary 2.3 and the method introduced in Section 2.2.3. In addition, if  $A_i$ ,  $1 \leq i \leq m$  is rank deficient, i.e.,  $\text{rank}(A_i) = r_i < n$ , then  $\text{rank}(\bar{A}_i) \leq 2nr_i - r_i^2$  according to Lemma 4.4.

While studying the single-parameter degree  $\deg_{\rho_i}(P(\rho))$ , all the parameters  $\rho_j$ ,  $j \neq i$  can be treated as constants. Therefore, according to Theorem 4.1, one has

$$\deg_{\rho_i}(P(\rho)) \leq 2nr_i - r_i^2. \quad (165)$$

Furthermore, if  $\dim(\mathcal{N}(A_1) \cap \mathcal{N}(A_2) \cap \dots \cap \mathcal{N}(A_m)) = r < n$ , then there exist linearly independent vectors  $v_1, v_2, \dots, v_r$ , such that

$$A_i v_j = 0, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, r. \quad (166)$$

Let  $V = [v_1, v_2, \dots, v_r]$ , and thus  $A_i V = 0$ ,  $i = 1, 2, \dots, m$ . Notice now that  $(A_i \otimes I + I \otimes A_i)(V \otimes V) = (A_i \otimes I)(V \otimes V) + (I \otimes A_i)(V \otimes V) = A_i V \otimes V + V \otimes A_i V = 0$ . Therefore,  $\bar{A}_i(V \otimes V) = 0$  for all  $i = 1, 2, \dots, m$ , which, according to Lemma 4.3, implies that

$$\dim(\mathcal{N}(\bar{A}_1) \cap \mathcal{N}(\bar{A}_2) \cap \dots \cap \mathcal{N}(\bar{A}_m)) \geq \text{rank}(V \otimes V) = \text{rank}(V) \text{rank}(V) = r^2$$

According to Lemma 6.3,

$$\deg(\text{Adj}(\bar{A}_0 + \rho_1 \bar{A}_1 + \rho_2 \bar{A}_2 + \dots + \rho_m \bar{A}_m)) \leq n^2 - r^2. \quad (167)$$

With (162) and (167), one has

$$\deg(P(\rho)) \leq n^2 - r^2.$$

■

**Remark 6.1** In case  $\Omega \subset \mathbb{R}^m$  is a connected set and if  $A(\rho)$  is Hurwitz for all  $\rho \in \Omega$ , it follows that

$$\text{sign}(\det(\bar{A}(\rho))) = \text{sign}(\det(\bar{A}(\rho'))), \quad \forall \rho, \rho' \in \Omega \quad (168)$$

This follows from the fact that  $\det(\bar{A}(\rho)) \neq 0$  for  $\rho \in \Omega$ , a connected subset in  $\mathbb{R}^m$ , and the continuity of the determinant of a matrix in terms of the matrix elements. Therefore,  $\text{sign}(\det(\bar{A}(\rho)))$  is independent of  $\rho$ . In this case Eq. (162) can be simplified to

$$\text{vec}(P) = \pm \text{Adj}(\bar{A}_0 + \rho_1 \bar{A}_1 + \rho_2 \bar{A}_2 + \dots + \rho_m \bar{A}_m) \text{vec}(I), \quad \forall \rho \in \Omega$$

Theorem 6.1 suggests a parameter-dependent Lyapunov matrix which can be used to test the stability of parameter-dependent LTI systems without conservatism.

In practice, it is convenient to test the stability of (148) with a polynomial Lyapunov matrix  $P(\rho)$  of the lowest possible degree in  $\rho$ , thus reducing the number of matrices in the polynomial expansion of  $P(\rho)$ .

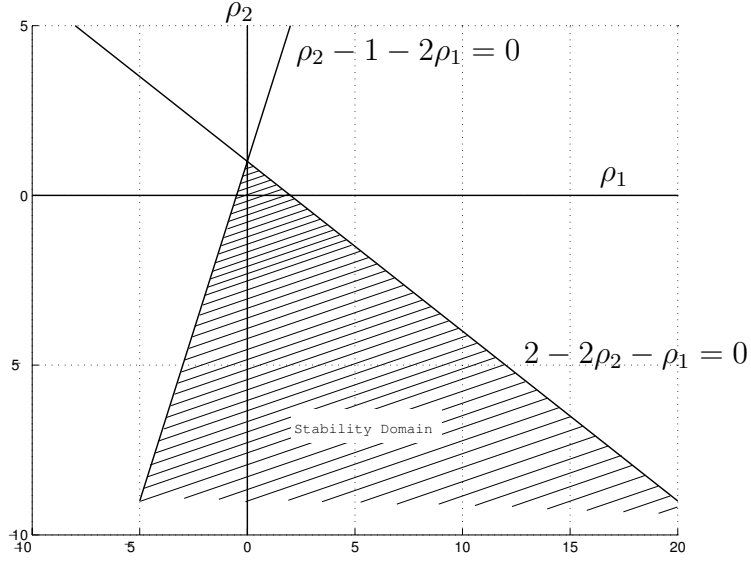
**Example 6.5** Consider the two parameter dependent matrix  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ , where

$$A(\rho) = \begin{bmatrix} -2 & 7 \\ 0 & -1 \end{bmatrix} + \rho_1 \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} + \rho_2 \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}.$$

It is clear that the stability domain for this  $A(\rho)$  is described as the following inequalities.

$$\lambda_1(A(\rho)) = -2 + \rho_1 + 2\rho_2 < 0$$

$$\lambda_2(A(\rho)) = -1 - 2\rho_1 + \rho_2 < 0$$



**Figure 8:** Whole Stability Domain

The set in  $\rho_1 - \rho_2$  space where both of these two inequalities are satisfied, is given in Fig. 8. A parameter-dependent Lyapunov matrix  $P(\rho)$  can be calculated according to Theorem 6.1 as follows,

$$P(\rho) = \begin{bmatrix} P_{11}(\rho) & * \\ P_{21}(\rho) & P_{22}(\rho) \end{bmatrix}$$

where

$$P_{11}(\rho) = 312 + 398\rho_1 - 684\rho_2 + 164\rho_1^2 - 604\rho_1\rho_2 + 486\rho_2^2 + 22\rho_1^3 - 128\rho_1^2\rho_2 + 224\rho_1\rho_2^2 - 114\rho_2^3$$

$$P_{21}(\rho) = 84 + 22\rho_1 - 216\rho_2 - 20\rho_1^2 - 50\rho_1\rho_2 + 180\rho_2^2 - 6\rho_1^3 + 14\rho_1^2\rho_2 + 28\rho_1\rho_2^2 - 48\rho_2^3$$

$$P_{22}(\rho) = 36 + 6\rho_1 - 108\rho_2 - 8\rho_1^2 - 12\rho_1\rho_2 + 108\rho_2^2 - 2\rho_1^3 + 8\rho_1^2\rho_2 + 6\rho_1\rho_2^2 - 36\rho_2^3$$

It can be easily checked that  $P(\rho) > 0$  and  $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$  if and only if  $\rho_1$  and  $\rho_2$  are inside the domain shown in Fig. 8.

### 6.2.2 Multi-Parameter Dependent Lyapunov Functions of Reduced Degree

In this section, we generalize the main result of Chapter 4, (Theorem 4.2) from the single-parameter LTIPD systems, to the multi-parameter LTIPD systems. As in Chapter 4,  $\overline{\text{vec}}(P)$ , the duplication matrix  $D_n$ , the lower Schlaeflian form  $\hat{A}$  along with their properties

will be used to find multi-parameter dependent Lyapunov functions for multi-parameter LTIPD systems.

From Definition 4.2 of  $\widehat{A}$ , it is clear that if  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2 + \dots + \rho_m A_m$  then

$$\widehat{A}(\rho) = \widehat{A}_0 + \rho_1 \widehat{A}_1 + \rho_2 \widehat{A}_2 + \dots + \rho_m \widehat{A}_m. \quad (169)$$

Similar to Corollary 4.1, the following property is immediate from (169) and Lemma 4.5.

**Corollary 6.1** *Given a parameter-dependent matrix  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2 + \dots + \rho_m A_m$ , which is Hurwitz for all  $\rho \in \Omega \in \mathbb{R}^m$ , then*

$$\det(\widehat{A}(\rho)) = \det(\widehat{A}_0 + \rho_1 \widehat{A}_1 + \rho_2 \widehat{A}_2 + \dots + \rho_m \widehat{A}_m) \neq 0, \quad \forall \rho \in \Omega \quad (170)$$

The following theorem can be used to show the existence of a parameter-dependent, polynomial Lyapunov matrix whenever  $A(\rho)$  is Hurwitz. The proof is similar to that of Theorem 4.2.

**Theorem 6.2** *Given matrices  $A_i \in \mathbb{R}^{n \times n}$ ,  $i = 0, 1, 2, \dots, m$ , and  $\rho = [\rho_1, \rho_2, \dots, \rho_m]^T \in \mathbb{R}^m$ , the following two statements are equivalent.*

(i)  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2 + \dots + \rho_m A_m$  is Hurwitz for all  $\rho \in \Omega$ .

(ii) There exist matrices  $P_{k_1, k_2, \dots, k_m} \in \mathbb{R}^{n \times n}$ , where  $k_i \in \mathbb{Z}_+^0$ ,  $i = 1, \dots, m$ , such that

$$A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0, \quad P(\rho) > 0, \quad \forall \rho \in \Omega \quad (171)$$

where,

$$P(\rho) = \sigma(\rho) \sum_{k_1+k_2+\dots+k_m \leq \frac{1}{2}n(n+1)-1} \rho_1^{k_1} \rho_2^{k_2} \dots \rho_m^{k_m} P_{k_1, k_2, \dots, k_m}, \quad (172)$$

$$\text{i.e.,} \quad \deg(P(\rho)) \leq \frac{1}{2}n(n+1)-1 \quad \forall \rho \in \Omega$$

where  $\sigma(\rho) = -\text{sign}(\det(\widehat{A}_0 + \rho_1 \widehat{A}_1 + \dots + \rho_m \widehat{A}_m))$ .

Furthermore, if  $A_i$  is rank deficient, i.e.,  $\text{rank}(A_i) = r_i < n$ ,  $1 \leq i \leq m$ , then,

$$\deg_{\rho_i}(P(\rho)) \leq \frac{1}{2}(2nr_i - r_i^2 + r_i). \quad (173)$$

*Proof.* [Theorem 6.2] (ii)  $\Rightarrow$  (i): It is obvious.

(i)  $\Rightarrow$  (ii): Since  $A(\rho)$  is Hurwitz for all  $\rho \in \Omega$ , from Corollary 6.1 it follows that  $\det(\hat{A}(\rho)) \neq 0$ . Choose the parameter-dependent, positive definite matrix

$$Q(\rho) = |\det(\hat{A}(\rho))|I_n > 0. \quad (174)$$

Since  $A(\rho)$  is Hurwitz for all  $\rho \in \Omega$ , the following Lyapunov equation has a unique, positive definite solution  $P(\rho) > 0$  [98]

$$A(\rho)P(\rho) + P(\rho)A^T(\rho) + |\det(\hat{A}(\rho))|I_n = 0 \quad (175)$$

Solving this Lyapunov equation, and following a similar approach as in the proof of Theorem 4.2, one has

$$\begin{aligned} (A(\rho) \oplus A(\rho))\text{vec}(P(\rho)) &= -|\det(\hat{A}(\rho))|\text{vec}(I_n) \\ \bar{A}(\rho)\text{vec}(P(\rho)) &= -|\det(\hat{A}(\rho))|\text{vec}(I_n) \\ D_n^+(\bar{A}(\rho))D_n\overline{\text{vec}}(P(\rho)) &= -|\det(\hat{A}(\rho))|\overline{\text{vec}}(I_n) \\ (\hat{A}(\rho))\overline{\text{vec}}(P(\rho)) &= -|\det(\hat{A}(\rho))|\overline{\text{vec}}(I_n) \\ \overline{\text{vec}}(P(\rho)) &= -|\det(\hat{A}(\rho))|(\hat{A}(\rho))^{-1}\overline{\text{vec}}(I_n) \end{aligned}$$

and thus,

$$\begin{aligned} \overline{\text{vec}}(P(\rho)) &= -|\det(\hat{A}(\rho))|\frac{\text{Adj}(\hat{A}(\rho))}{\det(\hat{A}(\rho))}\overline{\text{vec}}(I_n) \\ &= \sigma(\rho)\text{Adj}(\hat{A}_0 + \rho_1\hat{A}_1 + \rho_2\hat{A}_2 + \dots + \rho_m\hat{A}_m)\overline{\text{vec}}(I_n) \end{aligned} \quad (176)$$

where  $\sigma(\rho) := -\text{sign}(\det(\hat{A}_0 + \rho_1\hat{A}_1 + \rho_2\hat{A}_2 + \dots + \rho_m\hat{A}_m))$ .

Since  $\text{Adj}(\hat{A}(\rho))$  is a polynomial in  $\rho$ , and  $\hat{A}(\rho) \in \mathbb{R}^{\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)}$ , it follows from Lemma 6.3 that the polynomial degree of  $\text{Adj}(\hat{A}(\rho))$  satisfies

$$\deg\left(\text{Adj}(\hat{A}_0 + \rho_1\hat{A}_1 + \rho_2\hat{A}_2 + \dots + \rho_m\hat{A}_m)\right) \leq \frac{1}{2}n(n+1) - 1. \quad (177)$$

Therefore, there exist constant matrices  $N_{k_1, k_2, \dots, k_m}$  such that

$$\text{Adj}(\widehat{A}(\rho)) = \sum_{k_1=k_2=\dots=k_m=0}^{k_1+k_2+\dots+k_m \leq \frac{1}{2}n(n+1)-1} \rho_1^{k_1} \rho_2^{k_2} \dots \rho_m^{k_m} N_{k_1, k_2, \dots, k_m}. \quad (178)$$

The matrices  $N_{k_1, k_2, \dots, k_m}$  can be calculated with the method given in Corollary 2.3 or in Section 2.2.3. Since the mapping  $\overline{\text{vec}}(\cdot)$  is one-to-one, its inverse mapping  $\overline{\text{vec}}^{-1}(\cdot)$  exists. Therefore, (176) and (178) yield

$$P(\rho) = \sigma(\rho) \sum_{k_1=k_2=\dots=k_m=0}^{k_1+k_2+\dots+k_m \leq \frac{1}{2}n(n+1)-1} \rho_1^{k_1} \rho_2^{k_2} \dots \rho_m^{k_m} P_{k_1, k_2, \dots, k_m} \quad (179)$$

where  $P_{k_1, k_2, \dots, k_m}$  are constant matrices and  $P_{k_1, k_2, \dots, k_m} = \overline{\text{vec}}^{-1}(N_{k_1, k_2, \dots, k_m} \overline{\text{vec}}(I_n))$  for  $1 \leq i \leq m$  and  $0 \leq k_i \leq \frac{1}{2}n(n+1) - 1$ .

Furthermore, when  $A_i$ ,  $1 \leq i \leq m$  is rank deficient, i.e.,  $\text{rank}(A_i) = r_i < n$ , according to Lemma 4.6,  $\text{rank}(\widehat{A}_i) \leq \frac{1}{2}(2nr_i - r_i^2 + r_i)$ . While studying the single-parameter degree  $\deg_{\rho_i}(P(\rho))$ , all the parameters  $\rho_j$ ,  $j \neq i$  can be treated as constants and therefore Theorem 4.2 can be applied. It follows that  $\deg_{\rho_i}(P(\rho)) \leq \frac{1}{2}(2nr_i - r_i^2 + r_i)$ .  $\blacksquare$

**Remark 6.2** In case  $\Omega \subset \mathbb{R}^m$  is a connected set and if  $A(\rho)$  is Hurwitz for all  $\rho \in \Omega$ , it follows that

$$\text{sign}(\det(\widehat{A}(\rho))) = \text{sign}(\det(\widehat{A}(\rho'))), \quad \forall \rho, \rho' \in \Omega \quad (180)$$

This follows from the fact that  $\det(\widehat{A}(\rho)) \neq 0$  for  $\rho \in \Omega$ , a connected subset in  $\mathbb{R}^m$ , and the continuity of the determinant of a matrix in terms of the matrix elements. Therefore,  $\text{sign}(\det(\widehat{A}(\rho)))$  is independent of  $\rho$ . In this case (176) can be simplified to

$$\overline{\text{vec}}(P) = \pm \text{Adj}(\widehat{A}_0 + \rho_1 \widehat{A}_1 + \rho_2 \widehat{A}_2 + \dots + \rho_m \widehat{A}_m) \overline{\text{vec}}(I_n), \quad \forall \rho \in \Omega \quad (181)$$

**Example 6.6** The two-parameter dependent matrix  $A(\rho)$  in this example is the same as in Example 6.5. We study the stability of the matrix  $A(\rho)$  through the multi-parameter dependent Lyapunov function of reduced degree. Let

$$A(\rho) = \begin{bmatrix} -2 & 7 \\ 0 & -1 \end{bmatrix} + \rho_1 \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} + \rho_2 \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$$

It is clear that the stability domain for this  $A(\rho)$  is described as the following inequalities.

$$\begin{aligned}\lambda_1(A(\rho)) &= -2 + \rho_1 + 2\rho_2 < 0 \\ \lambda_2(A(\rho)) &= -1 - 2\rho_1 + \rho_2 < 0\end{aligned}$$

The set in  $\rho_1 - \rho_2$  space where both of these two inequalities are satisfied, is given in Fig. 8. A parameter-dependent Lyapunov matrix  $P(\rho)$  can be calculated according to Theorem 6.2 as follows,

$$P(\rho) = \begin{bmatrix} 104 + 98\rho_1 - 124\rho_2 + 22\rho_1^2 - 62\rho_1\rho_2 + 38\rho_2^2 & * \\ 28 - 2\rho_1 - 44\rho_2 - 6\rho_1^2 - 4\rho_1\rho_2 + 16\rho_2^2 & 12 - 2\rho_1 - 24\rho_2 - 2\rho_1^2 + 2\rho_1\rho_2 + 12\rho_2^2 \end{bmatrix}$$

It can be checked that  $P(\rho)$  satisfies  $P(\rho) > 0$  and  $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$  if and only if  $\rho_1$  and  $\rho_2$  are inside the domain shown in Fig. 8.

**Example 6.7** This example is taken from [63]. Consider the matrix  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ , where

$$A(\rho) = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix} + \rho_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \rho_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \quad (182)$$

The exact robust stability region for this problem is  $(-\infty, 1.75) \times (-\infty, 3)$  (see [63]). Reference [94] suggests an algorithm to calculate this stability domain. After calculating  $D_n$ ,  $\hat{A}$ , the Lyapunov matrix  $P(\rho) = P^T(\rho) \in \mathbb{R}^{3 \times 3}$  can be computed according to (172) or (181). Specifically,

$$\hat{A}_0 = \begin{bmatrix} -4 & 0 & -2 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & -1 & 0 \\ -1 & -1 & -6 & 0 & 0 & -1 \\ 0 & 0 & 0 & -6 & 0 & 0 \\ 0 & -1 & 0 & -1 & -7 & 0 \\ 0 & 0 & -2 & 0 & -2 & -8 \end{bmatrix}, \quad \hat{A}_1 = \begin{bmatrix} 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}.$$

Therefore, the multi-parameter dependent Lyapunov matrix  $P(\rho) = P^T(\rho) \in \mathbb{R}^{3 \times 3}$  is

calculated as follows

$$P(\rho) = \sigma(\rho) \begin{bmatrix} P_{11} & * & * \\ P_{21} & P_{22} & * \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

where

$$\begin{aligned} \sigma(\rho) &= -\text{sign}(\det(\hat{A}_0 + \rho_1 \hat{A}_1 + \rho_2 \hat{A}_2)) \\ &= -\text{sign}(1024\rho_2\rho_1^3 - 1920\rho_1^3 - 128\rho_2^2\rho_1^3 + 1568\rho_2^2\rho_1^2 - 9344\rho_2\rho_1^2 - 64\rho_2^3\rho_1^2 + 15648\rho_1^2 - 5232\rho_2^2\rho_1 \\ &\quad + 26656\rho_2\rho_1 + 304\rho_2^3\rho_1 - 41088\rho_1 - 336\rho_2^3 + 5040 * \rho_2^2 - 23520\rho_2 + 34272) \end{aligned}$$

and

$$\begin{aligned} P_{11} &= 8672\rho_1 + 6796\rho_2 - 2548\rho_1^2 + 12\rho_2^3\rho_1^2 + 24\rho_2^2\rho_1^3 - 1484\rho_2^2 + 1128\rho_2^2 * r - 292\rho_2^2\rho_1^2 \\ &\quad + 248\rho_2^3 + 1564\rho_2\rho_1^2 - 144\rho_2\rho_1^3 - 5624\rho_2\rho_1 + 100\rho_2^3 - 64\rho_2^3\rho_1 - 9828 \\ P_{21} &= 32\rho_1^3 - 184\rho_1^2 + 320\rho_1 - 32\rho_2\rho_1^3 + 184\rho_2\rho_1^2 - 320\rho_2\rho_1 - 168 - 168\rho_2 \\ P_{22} &= -64\rho_2\rho_1^3 + 320\rho_1^3 - 32\rho_2^2\rho_1^2 + 688\rho_2\rho_1^2 - 2608\rho_1^2 + 152\rho_2^2\rho_1 - 2160\rho_2\rho_1 + 6848\rho_1 - 168\rho_2^2 \\ &\quad + 2016\rho_2 - 5712 \\ P_{31} &= -4108\rho_1 - 1832\rho_2 + 2520 + 1836\rho_1^2 - 12\rho_2^3\rho_1^2 - 24\rho_2^2\rho_1^3 + 448\rho_2^2 - 676\rho_2^2\rho_1 + 252\rho_2^2\rho_1^2 - 248\rho_1^3 \\ &\quad - 1196\rho_2\rho_1^2 + 144\rho_2\rho_1^3 + 2884\rho_2\rho_1 - 32\rho_2^3 + 44\rho_2^3\rho_1 \\ P_{32} &= -32\rho_1^3 + 312\rho_1^2 - 928\rho_1 + 32\rho_2\rho_1^3 - 344\rho_2\rho_1^2 + 1080\rho_2\rho_1 + 840 + 1008\rho_2 + 32\rho_2^2\rho_1^2 \\ &\quad - 152\rho_2^2\rho_1 + 168\rho_2^2 \\ P_{33} &= 5744\rho_1 + 3860\rho_2 - 5124 + 2084\rho_1^2 + 12\rho_2^3\rho_1^2 + 24\rho_2^2\rho_1^3 - 1036\rho_2^2 + 984\rho_2^2\rho_1 - 276\rho_2^2\rho_1^2 \\ &\quad + 248\rho_1^3 + 1340\rho_2\rho_1^2 - 144\rho_2\rho_1^3 - 4048\rho_2\rho_1 + 92\rho_2^3 - 72\rho_2^3\rho_1, \end{aligned}$$

The eigenvalues of such a parameter-dependent Lyapunov matrix  $P(\rho)$  can be computed, and they are also parameter dependent. Since the eigenvalues are very complicated expressions of the parameters  $\rho_1$  and  $\rho_2$ , we check the positive definiteness of  $P(\rho)$  numerically. In fact, the following Lyapunov inequalities were checked.

$$P(\rho) > 0 \tag{183}$$

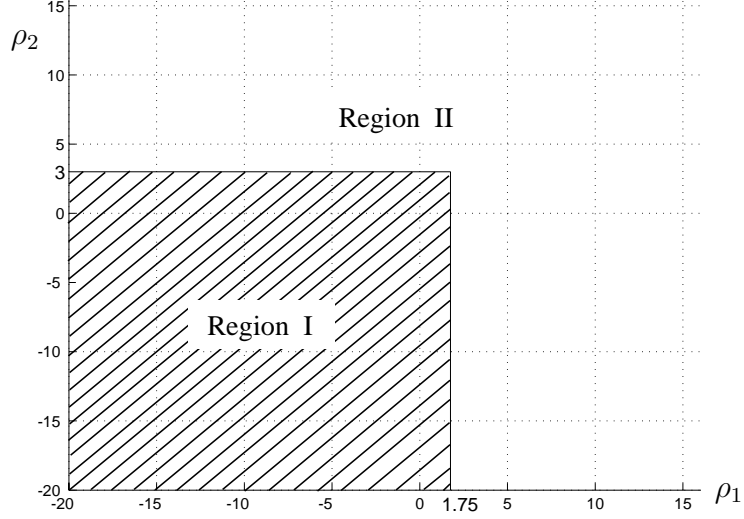
$$A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0 \tag{184}$$

The result is shown in Fig. 9. Two regions can be identified.

(i) Inside the Region I  $(-\infty, 1.75) \times (-\infty, 3)$  of Fig. 9, both (183) and (184) hold.



(ii) Outside the Region I  $(-\infty, 1.75) \times (-\infty, 3)$ , i.e., in Region II of Fig. 9, (184) always holds and (183) is never satisfied.



**Figure 9:** Lyapunov Inequalities Checking Result for Example 6.7.

### 6.2.3 Alternative Expressions of Parameter-Dependent Matrix-Valued Polynomials

Similarly to Chapter 4, the concept of the vector  $z^{[q]} \in \mathbb{R}^q$  as in Definition 4.3 is used in this section to offer an alternative description of the multi-parameter polynomial-type Lyapunov function in (172). Thus,  $P(\rho)$  will be expressed in the form of (186) as suggested in [13]. First, the function  $P(\rho) \in \mathbb{R}^{n \times n}$  in (172), for the parameter  $\rho$  inside a connected, stable domain, can be rewritten as follows,

$$P(\rho) = \sum_{k_1=\bar{k}_1, k_2=\bar{k}_2, \dots, k_m=\bar{k}_m} \rho_1^{k_1} \rho_2^{k_2} \cdots \rho_m^{k_m} P_{k_1, k_2, \dots, k_m}, \quad (185)$$

$$\text{i.e.,} \quad \deg_{\rho_i} (P(\rho)) \leq \bar{k}_i, \quad i = 1, 2, \dots, m$$

where  $\bar{k}_i = \frac{1}{2}(2nr_i - r_i^2 + r_i)$  if  $\text{rank}(A_i) = r_i < n$ , and  $\bar{k}_i = \frac{1}{2}n(n+1) - 1$  if  $\text{rank}(A_i) = r_i = n$  for  $1 \leq i \leq m$  since

$$\deg_{\rho_i}(P(\rho)) \leq \deg(P(\rho)) = \frac{1}{2}n(n+1) - 1 = \bar{k}_i.$$

Let  $\bar{\alpha}_i = \lceil \frac{\bar{k}_i}{2} \rceil + 1 \geq 2$ . Then the parameter dependent matrix in (185) can be expressed as

$$P(\rho) = \left( \rho_m^{[\bar{\alpha}_m]} \otimes \cdots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n \right)^T P_\Sigma \left( \rho_m^{[\bar{\alpha}_m]} \otimes \cdots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n \right) \quad (186)$$

where  $P_\Sigma$  is a symmetric matrix of size

$$(\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m \cdot n) \times (\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m \cdot n).$$

Moreover,  $P_\Sigma$  can be divided into  $(\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m) \times (\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m)$  blocks, where each block is an  $n \times n$  matrix. The matrix

$$\rho_m^{[\bar{\alpha}_m]} \otimes \cdots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n$$

is composed of  $\bar{\alpha}_1 \times \bar{\alpha}_2 \times \dots \times \bar{\alpha}_m$  blocks, where each block is an  $n \times n$  matrix.

**Definition 6.1** Given  $K = [\bar{\alpha}_1 \ \bar{\alpha}_2 \ \dots \ \bar{\alpha}_m]^T \in \mathbb{Z}_+^{0\ m}$ , the index function  $f_K$  is defined as

$$f_K(\alpha_1, \alpha_2, \dots, \alpha_m) = \bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_{m-1} \cdot \alpha_m + \bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_{m-2} \cdot \alpha_{m-1} + \dots + \bar{\alpha}_1 \cdot \alpha_2 + \alpha_1 + 1$$

The matrix-valued function  $P(\rho)$  in (185) can be rewritten in the form (186). It should be mentioned that the matrix  $P_\Sigma$  in (186) is not unique. One method to construct a possible  $P_\Sigma$  for the corresponding  $P(\rho) \in \mathbb{R}^{n \times n}$  is as follows.

(i) Let the index function  $f_K$  as in Definition 6.1, where

$$K = [\bar{\alpha}_1 \ \bar{\alpha}_2 \ \dots \ \bar{\alpha}_m]^T, \quad \bar{\alpha}_i = \lceil \frac{\deg_{\rho_i}(P(\rho))}{2} \rceil + 1 = \lceil \frac{\bar{k}_i}{2} \rceil + 1, \quad i = 1, 2, \dots, m.$$

(ii) Let  $\bar{P}_\Sigma$  be a square matrix with the same dimension of  $P_\Sigma$ , i.e.,

$$(\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m \cdot n) \times (\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m \cdot n).$$

$\bar{P}_\Sigma$  is also divided into  $(\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m) \times (\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m)$  blocks, and each block is an  $n \times n$  matrix.  $\bar{P}_{\Sigma, (i, j)}$  stands for the  $(i, j)$  block of  $\bar{P}_\Sigma$ . For every non-zero item

$\rho_1^{k_1} \rho_2^{k_2} \dots \rho_m^{k_m} P_{k_1, k_2, \dots, k_m}$  in (185), the  $(f_1, f_2)$  block of matrix  $\bar{P}_\Sigma$  is set to the value of  $P_{k_1, k_2, \dots, k_m}$ , i.e.,

$$\bar{P}_{\Sigma, (f_1, f_2)} = P_{k_1, k_2, \dots, k_m}.$$

where

$$\alpha_i = \lceil \frac{k_i}{2} \rceil, \quad \beta_i = \lfloor \frac{k_i}{2} \rfloor, \quad i = 1, 2, \dots, m$$

$$f_1 = f_K(\alpha_1, \alpha_2, \dots, \alpha_m), \quad f_2 = f_K(\beta_1, \beta_2, \dots, \beta_m) \quad .$$

Other blocks of matrix  $\bar{P}_\Sigma$ , which are not set to values, will be set to  $0_{n \times n}$ .

(iii)

$$P_\Sigma = \frac{1}{2}(\bar{P}_\Sigma + \bar{P}_\Sigma^T)$$

The matrix-valued function  $P(\rho)$  as in (186) with  $P_\Sigma$  constructed following this procedure is equivalent to the matrix-valued function  $P(\rho)$  as in (185).

**Example 6.8** Consider a two-parameter dependent matrix-valued function  $P(\rho) \in \mathbb{R}^{n \times n}$  as follows

$$P(\rho) = \sum_{k_1+k_2=2} \rho_1^{k_1} \rho_2^{k_2} P_{k_1, k_2} = P_{0,0} + \rho_1 P_{1,0} + \rho_2 P_{0,1} + \rho_1^2 P_{2,0} + \rho_1 \rho_2 P_{1,1} + \rho_2^2 P_{0,2}$$

Matrix  $P_\Sigma$  is sought to satisfy

$$P(\rho) = \left( \rho_2^{[2]} \otimes \rho_1^{[2]} \otimes I_n \right)^T P_\Sigma \left( \rho_2^{[2]} \otimes \rho_1^{[2]} \otimes I_n \right) \quad (187)$$

Two different matrices  $P_\Sigma$  that satisfy (187) are given below. The matrix  $P_\Sigma$  on the left was constructed according to the procedure (i) – (iii) above.

$$P_\Sigma = \begin{bmatrix} P_{0,0} & \frac{1}{2}P_{1,0} & \frac{1}{2}P_{0,1} & \frac{1}{2}P_{1,1} \\ \frac{1}{2}P_{1,0} & P_{2,0} & 0 & 0 \\ \frac{1}{2}P_{0,1} & 0 & P_{0,2} & 0 \\ \frac{1}{2}P_{1,1} & 0 & 0 & 0 \end{bmatrix}, \quad \text{or} \quad P_\Sigma = \begin{bmatrix} P_{0,0} & \frac{1}{2}P_{1,0} & \frac{1}{2}P_{0,1} & 0 \\ \frac{1}{2}P_{1,0} & P_{2,0} & \frac{1}{2}P_{1,1} & 0 \\ \frac{1}{2}P_{0,1} & \frac{1}{2}P_{1,1} & P_{0,2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Given a system matrix  $A(\rho)$  as in (148) and the matrix  $P(\rho)$  as in (186), let  $R(\rho) = A(\rho)P(\rho) + P(\rho)A^T(\rho)$ . We want to express  $R(\rho)$  in a form similar to (186).

Similarly to Chapter 4, let the matrices

$$\hat{J}_k = \begin{bmatrix} I_k & 0_{k \times 1} \end{bmatrix}, \quad \check{J}_k = \begin{bmatrix} 0_{k \times 1} & I_k \end{bmatrix}. \quad (188)$$

It is clear that  $\hat{J}_k z^{[k+1]} = z^{[k]}$ , and  $\check{J}_k z^{[k+1]} = z z^{[k]}$ . The following properties (shown in [13]) will be used in the sequel. Given  $A \in \mathbb{R}^{n \times n}$ ,  $M \in \mathbb{R}^{p \times q}$ , the following hold

- (i)  $(z^{[k]} \otimes I_p)M = (I_k \otimes M)(z^{[k]} \otimes I_q)$
- (ii)  $(z_m^{[k_m]} \otimes \dots \otimes z_1^{[k_1]} \otimes I_n)A = z_m^{[k_m]} \otimes \dots \otimes z_1^{[k_1]} \otimes A$   
 $= (I_{k_1 \cdot k_2 \dots k_m} \otimes A)(z_m^{[k_m]} \otimes \dots \otimes z_1^{[k_1]} \otimes I_n)$
- (iii)  $z_m^{[k_m]} \otimes \dots \otimes z_1^{[k_1]} \otimes I_n = (\hat{J}_{k_m} \otimes \dots \otimes \hat{J}_{k_1} \otimes I_n)(z_m^{[k_m+1]} \otimes \dots \otimes z_1^{[k_1+1]} \otimes I_n)$
- (iv)  $z_m^{[k_m]} \otimes \dots \otimes z_j z_j^{[k_j]} \otimes \dots \otimes z_1^{[k_1]} \otimes I_n$   
 $= (\hat{J}_{k_m} \otimes \dots \otimes \check{J}_{k_j} \otimes \dots \otimes \hat{J}_{k_1} \otimes I_n)(z_m^{[k_m+1]} \otimes \dots \otimes z_1^{[k_1+1]} \otimes I_n)$

**Lemma 6.4** *Given a matrix  $A(\rho) \in \mathbb{R}^{n \times n}$  as in (148) and a symmetric matrix  $P(\rho) \in \mathbb{R}^{n \times n}$  as in (186), let  $R(\rho) = A^T(\rho)P(\rho) + P(\rho)A(\rho)$ . Then*

$$R(\rho) = \left( \rho_m^{[\bar{\alpha}_m+1]} \otimes \dots \otimes \rho_1^{[\bar{\alpha}_1+1]} \otimes I_n \right)^T R_\Sigma \left( \rho_m^{[\bar{\alpha}_m+1]} \otimes \dots \otimes \rho_1^{[\bar{\alpha}_1+1]} \otimes I_n \right) \quad (189)$$

where

$$R_\Sigma = H_\Sigma^T P_\Sigma F_\Sigma + F_\Sigma^T P_\Sigma H_\Sigma \quad (190)$$

and where

$$H_\Sigma = \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes I_n \quad (191)$$

$$F_\Sigma = \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes A_0 + \sum_{i=1}^m \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_{i+1}} \otimes \check{J}_{\bar{\alpha}_i} \otimes \hat{J}_{\bar{\alpha}_{i-1}} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes A_i \quad (192)$$

*Proof.* From the expression of  $P(\rho)$  and  $A(\rho)$ , it follows that

$$\begin{aligned}
P(\rho)A(\rho) &= \left( \rho_m^{[\bar{\alpha}_m]} \otimes \dots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n \right)^T P_\Sigma \left( \rho_m^{[\bar{\alpha}_m]} \otimes \dots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n \right) A(\rho) \\
&= \left( \rho_m^{[\bar{\alpha}_m+1]} \otimes \dots \otimes \rho_1^{[\bar{\alpha}_1+1]} \otimes I_n \right)^T \left( \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes I_n \right)^T P_\Sigma \left( I_{\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m} \otimes A(\rho) \right) \\
&\quad \cdot \left( \rho_m^{[\bar{\alpha}_m]} \otimes \dots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n \right)
\end{aligned} \tag{193}$$

Since

$$\begin{aligned}
&\left( I_{\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m} \otimes A(\rho) \right) \left( \rho_m^{[\bar{\alpha}_m]} \otimes \dots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n \right) \\
&= \left( I_{\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m} \otimes A_0 \right) \left( \rho_m^{[\bar{\alpha}_m]} \otimes \dots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n \right) \\
&\quad + \left( I_{\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m} \otimes A_1 \right) \left( \rho_m^{[\bar{\alpha}_m]} \otimes \dots \otimes (\rho_1 \rho_1^{[\bar{\alpha}_1]}) \otimes I_n \right) \\
&\quad + \left( I_{\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m} \otimes A_2 \right) \left( \rho_m^{[\bar{\alpha}_m]} \otimes \dots \otimes (\rho_2 \rho_2^{[\bar{\alpha}_2]}) \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n \right) + \dots \\
&= \left[ \left( I_{\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m} \otimes A_0 \right) \left( \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes I_n \right) + \left( I_{\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m} \otimes A_1 \right) \left( \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \check{J}_{\bar{\alpha}_1} \otimes I_n \right) + \dots \right] \\
&\quad \cdot \left( \rho_m^{[\bar{\alpha}_m+1]} \otimes \dots \otimes \rho_1^{[\bar{\alpha}_1+1]} \otimes I_n \right)
\end{aligned} \tag{194}$$

and substituting (194) into (193), one obtains (189), (190) where

$$\begin{aligned}
H_\Sigma &= \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes I_n \\
F_\Sigma &= \left( I_{\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m} \otimes A_0 \right) \left( \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes I_n \right) \\
&\quad + \left( I_{\bar{\alpha}_1 \cdot \bar{\alpha}_2 \cdot \dots \cdot \bar{\alpha}_m} \otimes A_1 \right) \left( \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \check{J}_{\bar{\alpha}_1} \otimes I_n \right) + \dots \\
&= \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes A_0 + \sum_{i=1}^m \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_{i+1}} \otimes \check{J}_{\bar{\alpha}_i} \otimes \hat{J}_{\bar{\alpha}_{i-1}} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes A_i
\end{aligned}$$

This completes the proof. ■

**Example 6.9** Consider the matrices

$$\begin{aligned}
A(\rho) &= A_0 + \rho_1 A_1 + \rho_2 A_2 \in \mathbb{R}^{2 \times 2} \\
P(\rho) &= \left( \rho_2^{[2]} \otimes \rho_1^{[2]} \otimes I_2 \right)^T P_\Sigma \left( \rho_2^{[2]} \otimes \rho_1^{[2]} \otimes I_2 \right) \in \mathbb{R}^{2 \times 2}
\end{aligned}$$

where  $P_\Sigma \in \mathbb{R}^{8 \times 8}$ ,  $\bar{\alpha}_2 = \bar{\alpha}_1 = 2$ ,  $n = 2$ ,  $m = 2$ . It follows that

$$\begin{aligned}
R(\rho) &= A^T(\rho)P(\rho) + P(\rho)A(\rho) \\
&= \left( \rho_2^{[3]} \otimes \rho_1^{[3]} \otimes I_2 \right)^T R_\Sigma \left( \rho_2^{[3]} \otimes \rho_1^{[3]} \otimes I_2 \right)
\end{aligned}$$

where  $R_\Sigma \in \mathbb{R}^{18 \times 18}$  and

$$\begin{aligned}
R_\Sigma &= H_\Sigma^T P_\Sigma F_\Sigma + F_\Sigma^T P_\Sigma H_\Sigma \\
H_\Sigma &= \hat{J}_2 \otimes \hat{J}_2 \otimes I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{8 \times 18} \\
F_\Sigma &= \hat{J}_2 \otimes \hat{J}_2 \otimes A_0 + \check{J}_2 \otimes \check{J}_2 \otimes A_1 + \check{J}_2 \otimes \hat{J}_2 \otimes A_2 \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes A_0 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes A_1 \\
&\quad + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes A_2
\end{aligned}$$

It should be reminded that given  $A_i$ ,  $i = 1, 2, \dots, m$ ,  $R_\Sigma$  depends linearly on  $P_\Sigma$ .

### 6.3 LMI Conditions for Checking the Lyapunov Inequalities

In this section, we will express the inequalities  $P(\rho) > 0$  and  $R(\rho) < 0$  into LMIs without conservatism, where  $P(\rho)$  and  $R(\rho)$  are as in (186) and (189) respectively. The following lemma will be helpful in our developments.

**Lemma 6.5** ([41]) *Let matrices  $Q = Q^T$ ,  $F$ , and a compact subset of real matrices  $\mathcal{H}$  be given. The following statements are equivalent:*

(i) *for each  $H \in \mathcal{H}$*

$$\xi^T Q \xi < 0, \quad \forall \xi \neq 0 \text{ s.t. } HF\xi = 0$$

(ii) *there exist  $\Theta = \Theta^T$  s.t.*

$$Q + F^T \Theta F < 0, \quad N_H^T \Theta N_H \geq 0 \quad \forall H \in \mathcal{H}$$

According to the definition of  $\hat{J}_k$  and  $\check{J}_k$  as in (188), it is clear that

$$(\check{J}_k - \rho_i \hat{J}_k) \rho_i^{[k+1]} = 0 \quad \text{and} \quad (\check{J}_{k-1} - \rho_i \hat{J}_{k-1}) \rho_i^{[k]} = 0$$

**Definition 6.2** Given  $K = [k_1 \ k_2 \ \dots \ k_m]^T \in \mathbb{Z}_+^{0 \ m}$  and a positive integer  $n$ , let

$$\bar{C}_{K,i,j} = \alpha_{k_m} \otimes \alpha_{k_{m-1}} \otimes \dots \otimes \alpha_{k_{i+1}} \otimes \hat{J}_{k_{i-1}} \otimes \alpha_{k_{i-1}} \otimes \dots \otimes \alpha_{k_1} \otimes I_n \in \mathbb{R}^{m_1 \times n_1} \quad (195)$$

$$\bar{J}_{K,i,j} = \alpha_{k_m} \otimes \alpha_{k_{m-1}} \otimes \dots \otimes \alpha_{k_{i+1}} \otimes \check{J}_{k_{i-1}} \otimes \alpha_{k_{i-1}} \otimes \dots \otimes \alpha_{k_1} \otimes I_n \in \mathbb{R}^{m_1 \times n_1} \quad (196)$$

where  $m_1 = (k_1 - 1)(k_2 - 1) \dots (k_m - 1)n$ ,  $n_1 = k_1 k_2 \dots k_m n$ ,  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, 2^{m-1}$ . Every  $\alpha_{k_p}$  as in (195) and  $\alpha_{k_p}$  as in (196),  $p = 1, 2, \dots, m$ ,  $p \neq i$ , are same and can be either of the two possible values as follows,

$$\alpha_{k_p} = \hat{J}_{k_p-1}, \quad \text{or} \quad \alpha_{k_p} = \check{J}_{k_p-1}. \quad (197)$$

**Remark 6.3** Given  $K$  and  $i$  index pair in the Definition 6.2  $i = 1, 2, \dots, m$ , since every  $\alpha_{k_p}$ ,  $p = 1, 2, \dots, m$ ,  $p \neq i$  has two possibilities, therefore,  $\bar{C}_{K,i,\cdot}$  will have  $2^{m-1}$  possibilities. These possible values are arranged in sequence, and the  $j$ -th possible value is noted as  $\bar{C}_{K,i,j}$ ,  $j = 1, 2, \dots, 2^{m-1}$ . It is similar with  $\bar{J}_{K,i,j}$ ,  $j = 1, 2, \dots, 2^{m-1}$ .

**Definition 6.3** Given  $K = [k_1 \ k_2 \ \dots \ k_m]^T \in \mathbb{Z}_+^{0 \ m}$  and a positive integer  $n$ , matrices  $\mathcal{C}_K$ ,  $\mathcal{J}_K$  and  $\Delta_K$  are defined as follows,

$$\mathcal{C}_K = \begin{bmatrix} \bar{C}_{K,1,1} \\ \bar{C}_{K,1,2} \\ \vdots \\ \bar{C}_{K,i,j} \\ \vdots \\ \bar{C}_{K,m,2^{m-1}} \end{bmatrix}, \quad \mathcal{J}_K = \begin{bmatrix} \bar{J}_{K,1,1} \\ \bar{J}_{K,1,2} \\ \vdots \\ \bar{J}_{K,i,j} \\ \vdots \\ \bar{J}_{K,m,2^{m-1}} \end{bmatrix}, \quad \Delta_K = \begin{bmatrix} \rho_1 I_{m_1 \cdot 2^{m-1}} & & & \\ & \ddots & & \\ & & \rho_i I_{m_1 \cdot 2^{m-1}} & \\ & & & \ddots \\ & & & & \rho_m I_{m_1 \cdot 2^{m-1}} \end{bmatrix} \quad (198)$$

where  $i = 1, 2, \dots, m$ ,  $j = 1, 2, \dots, 2^{m-1}$  and  $\mathcal{C}_K, \mathcal{J}_K \in \mathbb{R}^{q \times n_1}$ ,  $\Delta_K \in \mathbb{R}^{q \times q}$  where  $m_1 = (k_1 - 1)(k_2 - 1) \dots (k_m - 1)n$ ,  $n_1 = k_1 k_2 \dots k_m n$  and  $q = m \cdot m_1 \cdot 2^{m-1}$ .

**Remark 6.4** The so defined matrices  $\mathcal{C}_K$ ,  $\mathcal{J}_K$  and  $\Delta_K$  have a special property, which will be studied in Lemma 6.6. This property will be used in Theorem 6.3.

**Example 6.10** Given the vector  $K = [k_1, k_2]^T = [2, 2]^T$  and a positive integer  $n = 2$  and according to the Definition 6.3, we construct the matrices  $\mathcal{C}_K$ ,  $\mathcal{J}_K$  and  $\Delta_K$  corresponding

to this  $K$  and  $n$  as follows,

$$\mathcal{C}_K = \begin{bmatrix} \bar{C}_{K,1,1} \\ \bar{C}_{K,1,2} \\ \bar{C}_{K,2,1} \\ \bar{C}_{K,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix}, \quad \mathcal{J}_K = \begin{bmatrix} \bar{J}_{K,1,1} \\ \bar{J}_{K,1,2} \\ \bar{J}_{K,2,1} \\ \bar{J}_{K,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix}, \quad \Delta_K = \begin{bmatrix} \rho_1 I_4 & \\ & \rho_2 I_4 \end{bmatrix}. \quad (199)$$

It can be easily checked that

$$(\mathcal{J}_K - \Delta_K \mathcal{C}_K)(\rho_m^{[2]} \otimes \rho_1^{[2]} \otimes I_2) = 0$$

We now give a special property of matrices  $\mathcal{C}_K$ ,  $\mathcal{J}_K$  and  $\Delta_K$ .

**Lemma 6.6** *For a sequence of  $\rho_i^{[k_i]}$ ,  $i = 1, 2, \dots, m$ , let  $K = [k_1 \ k_2 \ \dots \ k_m]^T \in \mathbb{Z}^m$ . Then*

$$(\mathcal{J}_K - \Delta_K \mathcal{C}_K)(\rho_m^{[k_m]} \otimes \dots \otimes \rho_1^{[k_1]} \otimes I_n) = 0$$

*Proof.* Notice that  $\mathcal{J}_K - \Delta_K \mathcal{C}_K$  has the form

$$\mathcal{J}_K - \Delta_K \mathcal{C}_K = \begin{bmatrix} \bar{J}_{K,1,1} - \rho_1 \bar{C}_{K,1,1} \\ \bar{J}_{K,1,2} - \rho_1 \bar{C}_{K,1,2} \\ \vdots \\ \bar{J}_{K,i,j} - \rho_i \bar{C}_{K,i,j} \\ \vdots \end{bmatrix}$$



Let us study just one ‘row’ of the above matrix.

$$\begin{aligned}
& (\bar{J}_{K,i,j} - \rho_i \bar{C}_{K,i,j})(\rho_m^{[k_m]} \otimes \cdots \otimes \rho_1^{[k_1]} \otimes I_n) \\
&= \left\{ \alpha_{k_m} \otimes \alpha_{k_{m-1}} \otimes \cdots \otimes \alpha_{k_{i+1}} \otimes (\check{J}_{k_i-1} - \rho_i \hat{J}_{k_i-1}) \otimes \alpha_{k_{i-1}} \otimes \cdots \otimes \alpha_{k_1} \otimes I_n \right\} \\
&\quad \times (\rho_m^{[k_m]} \otimes \cdots \otimes \rho_1^{[k_1]} \otimes I_n) \\
&= \left\{ [\alpha_{k_m} \otimes \alpha_{k_{m-1}} \otimes \cdots \otimes \alpha_{k_{i+1}} \otimes (\check{J}_{k_i-1} - \rho_i \hat{J}_{k_i-1})] \otimes [\alpha_{k_{i-1}} \otimes \cdots \otimes \alpha_{k_1} \otimes I_n] \right\} \\
&\quad \left\{ [\rho_m^{[k_m]} \otimes \cdots \otimes \rho_i^{[k_i]}] \otimes [\rho_{i-1}^{[k_{i-1}]} \otimes \cdots \otimes \rho_1^{[k_1]} \otimes I_n] \right\} \\
&= \left\{ [\alpha_{k_m} \otimes \alpha_{k_{m-1}} \otimes \cdots \otimes \alpha_{k_{i+1}} \otimes (\check{J}_{k_i-1} - \rho_i \hat{J}_{k_i-1})] [\rho_m^{[k_m]} \otimes \cdots \otimes \rho_i^{[k_i]}] \right\} \\
&\quad \otimes \left\{ [\alpha_{k_{i-1}} \otimes \cdots \otimes \alpha_{k_1} \otimes I_n] [\rho_{i-1}^{[k_{i-1}]} \otimes \cdots \otimes \rho_1^{[k_1]} \otimes I_n] \right\} \\
&= \left\{ [(\alpha_{k_m} \otimes \alpha_{k_{m-1}} \otimes \cdots \otimes \alpha_{k_{i+1}})(\rho_m^{[k_m]} \otimes \cdots \otimes \rho_{i+1}^{[k_{i+1}]})] \otimes [(\check{J}_{k_i-1} - \rho_i \hat{J}_{k_i-1})\rho_i^{[k_i]}] \right\} \\
&\quad \otimes \left\{ [\alpha_{k_{i-1}} \otimes \cdots \otimes \alpha_{k_1} \otimes I_n] [\rho_{i-1}^{[k_{i-1}]} \otimes \cdots \otimes \rho_1^{[k_1]} \otimes I_n] \right\} \\
&= \left\{ [(\alpha_{k_m} \otimes \alpha_{k_{m-1}} \otimes \cdots \otimes \alpha_{k_{i+1}})(\rho_m^{[k_m]} \otimes \cdots \otimes \rho_{i+1}^{[k_{i+1}]})] \otimes 0 \right\} \\
&\quad \otimes \left\{ [\alpha_{k_{i-1}} \otimes \cdots \otimes \alpha_{k_1} \otimes I_n] [\rho_{i-1}^{[k_{i-1}]} \otimes \cdots \otimes \rho_1^{[k_1]} \otimes I_n] \right\} \\
&= 0
\end{aligned} \tag{200}$$

Therefore,  $(\mathcal{J}_K - \Delta_K \mathcal{C}_K)(\rho_m^{[k_m]} \otimes \cdots \otimes \rho_1^{[k_1]} \otimes I_n) = 0$  ■

For  $P(\rho)$  in (186),  $P(\rho) > 0$  is equivalent to the condition that for any  $x \in \mathbb{R}^n$ ,  $x^T P(\rho)x > 0$ , i.e.,

$$x^T (\rho_m^{[\bar{\alpha}_m]} \otimes \cdots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n)^T P_\Sigma (\rho_m^{[\bar{\alpha}_m]} \otimes \cdots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n) x > 0$$

Let  $\xi = (\rho_m^{[\bar{\alpha}_m]} \otimes \cdots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n)x$ , then  $P(\rho) > 0$  is equivalent to

$$\xi^T P_\Sigma \xi > 0 \quad \text{s.t.} \quad (\mathcal{J}_K - \Delta_K \mathcal{C}_K)\xi = 0$$

**Theorem 6.3** *The following two statements are equivalent:*

(i) For  $\rho = [\rho_1, \rho_2, \dots, \rho_m]^T \in [-1, +1]^m$ ,

$$P(\rho) = (\rho_m^{[\bar{\alpha}_m]} \otimes \cdots \otimes \rho_2^{[\bar{\alpha}_2]} \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n)^T P_\Sigma (\rho_m^{[\bar{\alpha}_m]} \otimes \cdots \otimes \rho_2^{[\bar{\alpha}_2]} \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n) > 0.$$

(ii) There exist matrices  $D_1, D_2, \dots, D_m \in \mathbb{R}^{q \times q}$ ,  $D_1 > 0, D_2 > 0, \dots, D_m > 0$ , where

$m_1 = (\bar{\alpha}_1 - 1)(\bar{\alpha}_2 - 1) \dots (\bar{\alpha}_m - 1)n$  and  $q = m_1 \cdot 2^{m-1}$ , such that

$$-P_\Sigma + \begin{bmatrix} \mathcal{J}_K \\ \mathcal{C}_K \end{bmatrix}^T \begin{bmatrix} -D_1 & & & & \\ & \ddots & & & \\ & & -D_m & & \\ & & & D_1 & \\ & & & & \ddots \\ & & & & & D_m \end{bmatrix} \begin{bmatrix} \mathcal{J}_K \\ \mathcal{C}_K \end{bmatrix} < 0, \quad (201)$$

where  $\mathcal{J}_K$  and  $\mathcal{C}_K$  are defined as (198) corresponding to  $K = [\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_m]$  and  $n$ .

**Remark 6.5** Theorem 6.3 can be used to check the positive definiteness of a multi-parameter dependent Lyapunov function over the compact set  $[-1, +1]^m$ . The result in [95] is used to check the positiveness of a single-parameter dependent Lyapunov function over the compact set  $[-1, +1]$ , which is a special case  $m = 1$  of Theorem 6.3.

*Proof.* [Theorem 6.3] Notice that  $P(\rho) > 0$  is equivalent to the condition that, for any  $x \in \mathbb{R}^n$ , there is  $x^T P(\rho)x > 0$  for  $\rho \in [-1, +1]^m$ . Let

$$\xi = (\rho_m^{[\bar{\alpha}_m]} \otimes \dots \otimes \rho_2^{[\bar{\alpha}_2]} \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n)x, \quad (202)$$

then,  $(\mathcal{J}_K - \Delta_K \mathcal{C}_K)\xi = 0$ ,  $\mathcal{J}_K$ ,  $\Delta_K$  and  $\mathcal{C}_K$  are defined as in (198). Therefore,  $P(\rho) > 0$  for  $\rho \in [-1, +1]^m$  is equivalent to

$$-\xi^T P_\Sigma \xi < 0, \quad \text{s.t.} \quad HF\xi = 0, \quad (203)$$

where

$$H = [I, -\Delta_K], \quad F = \begin{bmatrix} \mathcal{J}_K \\ \mathcal{C}_K \end{bmatrix}, \quad \text{and thus} \quad \mathcal{N}_H = \begin{bmatrix} \Delta_K \\ I \end{bmatrix}.$$

According to Lemma 6.5, inequality (203) is equivalent to the existence of a matrix  $\Theta = \Theta^T$ , such that

$$-P_\Sigma + \begin{bmatrix} \mathcal{J}_K \\ \mathcal{C}_K \end{bmatrix}^T \Theta \begin{bmatrix} \mathcal{J}_K \\ \mathcal{C}_K \end{bmatrix} < 0 \quad (204)$$

and

$$\begin{bmatrix} \Delta_K \\ I \end{bmatrix}^T \Theta \begin{bmatrix} \Delta_K \\ I \end{bmatrix} \geq 0 \quad (205)$$

Condition  $\rho_i \in [-1, +1]$ ,  $i = 1, 2, \dots, m$  is equivalent to the existence of matrices  $D_1, D_2, \dots, D_m \in \mathbb{R}^{q \times q}$ ,  $D_1 > 0, D_2 > 0, \dots, D_m > 0$ , where  $m_1 = (k_1 - 1)(k_2 - 1) \dots (k_m - 1)n$  and  $q = m_1 \cdot 2^{m-1}$ , such that

$$\begin{bmatrix} (1 - \rho_1^2)D_1 & & & \\ & (1 - \rho_2^2)D_2 & & \\ & & \ddots & \\ & & & (1 - \rho_m^2)D_m \end{bmatrix} \geq 0, \quad \text{i.e.,} \quad (206)$$

$$\begin{bmatrix} \Delta_K \\ I \end{bmatrix}^T \begin{bmatrix} -D_1 & & & \\ & \ddots & & \\ & & -D_m & \\ & & & D_1 & \\ & & & & \ddots \\ & & & & & D_m \end{bmatrix} \begin{bmatrix} \Delta_K \\ I \end{bmatrix} \geq 0$$

In summary, the condition that  $P(\rho) > 0$  for the parameter  $\rho \in [-1, +1]^m$  is equivalent to the condition that matrix inequalities (204) and (205) hold. Since  $\rho \in [-1, +1]^m$ , the condition that matrix inequality (205) holds is equivalent to the condition that matrix inequality (206) holds. Therefore, inequality (204) can be rewritten as (201).  $\blacksquare$

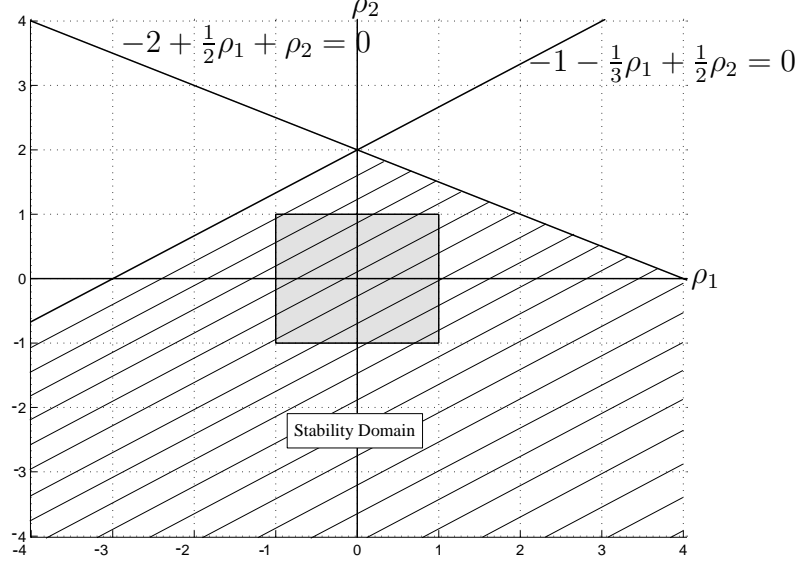
**Example 6.11** Consider the two-parameter dependent matrix  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$  where

$$A(\rho) = \begin{bmatrix} -2 & 7 \\ 0 & -1 \end{bmatrix} + \rho_1 \begin{bmatrix} \frac{1}{2} & 3 \\ 0 & -\frac{1}{3} \end{bmatrix} + \rho_2 \begin{bmatrix} 1 & -4 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

It is clear that the stability domain for this  $A(\rho)$  is described as the following inequalities.

$$\begin{aligned} \lambda_1(A(\rho)) &= -2 + \frac{1}{2}\rho_1 + \rho_2 < 0 \\ \lambda_2(A(\rho)) &= -1 - \frac{1}{3}\rho_1 + \frac{1}{2}\rho_2 < 0 \end{aligned}$$

The set in  $\rho_1 - \rho_2$  space where both of these two inequalities are satisfied, is given in Fig. 10.



**Figure 10:** Whole Stability Domain

According to Theorem 6.2, the condition that  $A(\rho)$  is Hurwitz for  $\rho \in [-1, +1] \times [-1, +1]$  is equivalent to the condition that the following parameter-dependent matrix  $P(\rho)$  is positive definite for all  $\rho \in [-1, +1] \times [-1, +1]$ .

$$\begin{aligned}
 P(\rho) &= -\overline{\text{vec}}^{-1} \left( |\det(\hat{A}(\rho))| \frac{1}{\det(\hat{A}(\rho))} \text{Adj}(\hat{A}(\rho)) \overline{\text{vec}}(I_2) \right) \\
 &= \begin{bmatrix} 6 & 14 \\ 14 & 110 \end{bmatrix} + \rho_1 \begin{bmatrix} \frac{5}{3} & \frac{32}{3} \\ \frac{32}{3} & \frac{241}{3} \end{bmatrix} + \rho_2 \begin{bmatrix} -6 & -15 \\ -15 & -124 \end{bmatrix} \\
 &\quad + \rho_1^2 \begin{bmatrix} -\frac{1}{9} & 2 \\ 2 & \frac{109}{6} \end{bmatrix} + \rho_2^2 \begin{bmatrix} \frac{3}{2} & 4 \\ 4 & 35 \end{bmatrix} + \rho_1 \rho_2 \begin{bmatrix} -\frac{5}{6} & -\frac{17}{3} \\ -\frac{17}{3} & -\frac{277}{6} \end{bmatrix}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
P(\rho) &= P_{0,0} + \rho_1 P_{1,0} + \rho_2 P_{0,1} + \rho_1^2 P_{2,0} + \rho_2^2 P_{0,2} + \rho_1 \rho_2 P_{1,1} \\
&= \left( \rho_2^{[2]} \otimes \rho_1^{[2]} \otimes I \right)^T P_\Sigma \left( \rho_2^{[2]} \otimes \rho_1^{[2]} \otimes I \right) \\
&= \left( \rho_2^{[2]} \otimes \rho_1^{[2]} \otimes I \right)^T \begin{bmatrix} P_{0,0} & \frac{1}{2}P_{1,0} & \frac{1}{2}P_{0,1} & 0 \\ \frac{1}{2}P_{1,0} & P_{2,0} & \frac{1}{2}P_{1,1} & 0 \\ \frac{1}{2}P_{0,1} & \frac{1}{2}P_{1,1} & P_{0,2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left( \rho_2^{[2]} \otimes \rho_1^{[2]} \otimes I \right). \quad (207)
\end{aligned}$$

We now apply Theorem 6.3 to check whether  $P(\rho)$  is positive definite for all  $\rho \in [-1, +1] \times [-1, +1]$ . For the parameter-dependent matrix  $P(\rho)$  in (207),  $K = [2, 2]$ ,  $\mathcal{C}_K$  and  $\mathcal{J}_K$  are listed as follows,

$$\mathcal{C}_K = \begin{bmatrix} \bar{C}_{K,1,1} \\ \bar{C}_{K,1,2} \\ \bar{C}_{K,2,1} \\ \bar{C}_{K,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix}, \quad \mathcal{J}_K = \begin{bmatrix} \bar{J}_{K,1,1} \\ \bar{J}_{K,1,2} \\ \bar{J}_{K,2,1} \\ \bar{J}_{K,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix}.$$

Their numerical values are

$$\mathcal{C}_K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{J}_K = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

One possible solution  $D_1, D_2$  for LMI (201) is as follows,

$$D_1 = \begin{bmatrix} 3.0166 & 5.7212 & -2.0825 & -4.7455 \\ 5.7212 & 44.3803 & -5.0404 & -39.4862 \\ -2.0825 & -5.0404 & 2.0696 & 5.0789 \\ -4.7455 & -39.4862 & 5.0789 & 40.9014 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1.7816 & 4.3719 & 0.5522 & 3.3481 \\ 4.3719 & 34.0496 & 3.5149 & 24.6075 \\ 0.5522 & 3.5149 & 1.2206 & 3.7040 \\ 3.3481 & 24.6075 & 3.7040 & 28.4463 \end{bmatrix}$$

Therefore, this  $P(\rho) > 0$  for  $\rho \in [-1, +1] \times [-1, +1]$  and thus  $A(\rho)$  is Hurwitz for  $\rho \in [-1, +1] \times [-1, +1]$ . This example agrees with Theorem 6.3.

**Theorem 6.4** Given matrices  $A_i \in \mathbb{R}^{n \times n}$ ,  $i = 0, 1, 2, \dots, m$ , and  $\rho = [\rho_1, \rho_2, \dots, \rho_m]^T \in \mathbb{R}^m$ , let

$$\bar{k} = \frac{1}{2}n(n+1) - 1, \quad \bar{\alpha} = \lceil \frac{\bar{k}}{2} \rceil + 1, \quad m_1 = (\bar{\alpha} - 1)^m n, \quad m_2 = \bar{\alpha}^m n$$

$$K_1 = [\bar{\alpha}, \bar{\alpha}, \dots, \bar{\alpha}]^T \in \mathbb{R}^m, \quad K_2 = [(\bar{\alpha} + 1), (\bar{\alpha} + 1), \dots, (\bar{\alpha} + 1)]^T \in \mathbb{R}^m.$$

Then, the following two statements are equivalent.

(i)  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2 + \dots + \rho_m A_m$  is Hurwitz for all  $\rho \in [-1, +1]^m$ .

(ii) There exist symmetric matrix  $P_\Sigma \in \mathbb{R}^{m_2 \times m_2}$ , positive definite matrices  $D_{P,1}, D_{P,2}, \dots, D_{P,m} \in \mathbb{R}^{m_1 \cdot 2^{m-1} \times m_1 \cdot 2^{m-1}}$ , and positive definite matrices  $D_{R,1}, D_{R,2}, \dots, D_{R,m} \in \mathbb{R}^{m_2 \cdot 2^{m-1} \times m_2 \cdot 2^{m-1}}$ , such that

$$-P_\Sigma + \begin{bmatrix} \mathcal{J}_{K_1} \\ \mathcal{C}_{K_1} \end{bmatrix}^T \begin{bmatrix} -D_{P,1} & & & \\ & \ddots & & \\ & & -D_{P,m} & \\ & & & D_{P,1} \\ & & & & \ddots \\ & & & & & D_{P,m} \end{bmatrix} \begin{bmatrix} \mathcal{J}_{K_1} \\ \mathcal{C}_{K_1} \end{bmatrix} < 0 \quad (208)$$

$$H_\Sigma^T P_\Sigma F_\Sigma + F_\Sigma^T P_\Sigma H_\Sigma + \begin{bmatrix} \mathcal{J}_{K_2} \\ \mathcal{C}_{K_2} \end{bmatrix}^T \begin{bmatrix} -D_{R,1} & & & \\ & \ddots & & \\ & & -D_{R,m} & \\ & & & D_{R,1} \\ & & & & \ddots \\ & & & & & D_{R,m} \end{bmatrix} \begin{bmatrix} \mathcal{J}_{K_2} \\ \mathcal{C}_{K_2} \end{bmatrix} < 0 \quad (209)$$

where

$$H_\Sigma = \hat{J}_{\bar{\alpha}}^{m \otimes} \otimes I_n$$

$$F_\Sigma = \hat{J}_{\bar{\alpha}}^{m \otimes} \otimes A_0 + \sum_{i=1}^m \hat{J}_{\bar{\alpha}}^{(m-i) \otimes} \otimes \check{J}_{\bar{\alpha}} \otimes \hat{J}_{\bar{\alpha}}^{(i-1) \otimes} \otimes A_i.$$

The matrices  $\mathcal{J}_{K_1}, \mathcal{C}_{K_1}$  are constructed according to Definition 6.3 and the vector  $K_1$ , and the matrices  $\mathcal{J}_{K_2}, \mathcal{C}_{K_2}$  are constructed according to Definition 6.3 and the vector  $K_2$ .

**Corollary 6.2** In Theorem 6.4, if  $\text{rank}(A_i) = r_i < n$ ,  $1 \leq i \leq m$ , the size of LMIs (208)

and (209) can be reduced. The constant matrices  $H_\Sigma$  and  $F_\Sigma$  will be

$$H_\Sigma = \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes I_n$$

$$F_\Sigma = \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes A_0 + \sum_{i=1}^m \hat{J}_{\bar{\alpha}_m} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_{i+1}} \otimes \check{J}_{\bar{\alpha}_i} \otimes \hat{J}_{\bar{\alpha}_{i-1}} \otimes \dots \otimes \hat{J}_{\bar{\alpha}_1} \otimes A_i$$

where  $\bar{\alpha}_i = \lceil \frac{\bar{k}_i}{2} \rceil + 1$  and where  $\bar{k}_i = \frac{1}{2}(2nr_i - r_i^2 + r_i)$ . The size of matrices  $D_{P,1}, D_{P,2}, \dots, D_{P,m}$  can be reduced to  $q \times q$  where  $q = m_1 \cdot 2^{m-1}$  and where  $m_1 = (\prod_{i=1}^m (\bar{\alpha}_i - 1))n$ . The size of matrices  $D_{R,1}, D_{R,2}, \dots, D_{R,m}$  can be reduced to  $d \times d$  where  $d = m_2 \cdot 2^{m-1}$  and where  $m_2 = (\prod_{i=1}^m \bar{\alpha}_i)n$ . The size of matrices  $\mathcal{J}_{K_1}$  and  $\mathcal{J}_{C_1}$  can be also reduced and they will be constructed according to Definition 6.3 and the vector  $K_1 = [\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_m]^T$ . The size of matrices  $\mathcal{J}_{K_2}$  and  $\mathcal{J}_{C_2}$  can be also reduced and they will be constructed according to Definition 6.3 and the vector  $K_2 = [(\bar{\alpha}_1 + 1), (\bar{\alpha}_2 + 1), \dots, (\bar{\alpha}_m + 1)]^T$ .

*Proof.* [Of Theorem 6.4] According to Theorem 6.2,  $A(\rho)$  is Hurwitz for  $\rho \in [-1, +1]^m$  if and only if there exists a polynomial Lyapunov function  $P(\rho)$  of degree  $\deg_{\rho_i}(P(\rho)) \leq \frac{1}{2}n(n+1) - 1$  as in (172) such that

$$P(\rho) > 0, \quad R(\rho) = A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0.$$

This  $P(\rho)$  in (172) can be expressed in (186), the corresponding  $R(\rho)$  can be expressed in (189) according to Lemma 6.4.

The condition  $P(\rho) > 0$ , where  $P(\rho)$  is in form of (186) and  $\rho \in [-1, +1]^m$ , is equivalent to the condition that LMI (208) holds according to Theorem 6.3. Similarly, the condition that  $R(\rho) < 0$ , where  $R(\rho)$  is in form of (189) and  $\rho \in [-1, +1]^m$ , is equivalent to the condition that LMI (209) holds. ■

*Proof.* [Of Corollary 6.2] Following the similar procedure to the proof of Theorem 6.4, one can easily prove Corollary 6.2 with the fact that if  $\text{rank}(A_i) = r_i < n$ ,  $1 \leq i \leq m$ , then,  $\deg_{\rho_i}(P(\rho)) \leq \frac{1}{2}(2nr_i - r_i^2 + r_i)$ . ■

The following numerical example shows one application of Theorem 6.4.

**Example 6.12** Consider again the two-parameter dependent matrix  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$  as in Example 6.11, where

$$A(\rho) = \begin{bmatrix} -2 & 7 \\ 0 & -1 \end{bmatrix} + \rho_1 \begin{bmatrix} \frac{1}{2} & 3 \\ 0 & -\frac{1}{3} \end{bmatrix} + \rho_2 \begin{bmatrix} 1 & -4 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

The set in  $\rho_1 - \rho_2$  space where  $A(\rho)$  is Hurwitz is shown in Fig. 10. It is clear that  $A(\rho)$  is Hurwitz for  $\rho \in [-1, +1] \times [-1, +1]$ . We now apply Theorem 6.4 to test whether  $A(\rho)$  is Hurwitz for  $\rho \in [-1, +1] \times [-1, +1]$ . For this specific problem,

$$n = 2, \quad m = 2, \quad \bar{k} = \frac{1}{2}n(n+1) - 1 = 2, \quad \bar{\alpha} = \lceil \frac{\bar{k}}{2} \rceil + 1 = 2,$$

$$m_1 = (\bar{\alpha} - 1)^m n = 2, \quad m_2 = \bar{\alpha}^m n = 8$$

$$K_1 = [\bar{\alpha}, \bar{\alpha}]^T = [2, 2]^T, \quad K_2 = [(\bar{\alpha} + 1), (\bar{\alpha} + 1)]^T = [3, 3]^T,$$

$$P_\Sigma \in \mathbb{R}^{8 \times 8}, D_{P,1}, D_{P,2} \in \mathbb{R}^{4 \times 4}, D_{R,1}, D_{R,2} \in \mathbb{R}^{16 \times 16}$$

and

$$\begin{aligned} \mathcal{C}_{K_1} &= \begin{bmatrix} \bar{C}_{K_1,1,1} \\ \bar{C}_{K_1,1,2} \\ \bar{C}_{K_1,2,1} \\ \bar{C}_{K_1,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix}, \quad \mathcal{J}_{K_1} = \begin{bmatrix} \bar{J}_{K_1,1,1} \\ \bar{J}_{K_1,1,2} \\ \bar{J}_{K_1,2,1} \\ \bar{J}_{K_1,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix}, \\ \mathcal{C}_{K_2} &= \begin{bmatrix} \bar{C}_{K_2,1,1} \\ \bar{C}_{K_2,1,2} \\ \bar{C}_{K_2,2,1} \\ \bar{C}_{K_2,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \hat{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \hat{J}_2 \otimes \check{J}_2 \otimes I_2 \end{bmatrix}, \quad \mathcal{J}_{K_2} = \begin{bmatrix} \bar{J}_{K_2,1,1} \\ \bar{J}_{K_2,1,2} \\ \bar{J}_{K_2,2,1} \\ \bar{J}_{K_2,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_2 \otimes \check{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \check{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \check{J}_2 \otimes I_2 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} H_\Sigma &= \hat{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ F_\Sigma &= \hat{J}_2 \otimes \hat{J}_2 \otimes A_0 + \hat{J}_2 \otimes \check{J}_2 \otimes A_1 + \check{J}_2 \otimes \hat{J}_2 \otimes A_2. \end{aligned}$$



One possible solution for LMI (208) and (209) is given by

$$P_{\Sigma} = \begin{bmatrix} 27.9276 & 51.0744 & -0.7887 & 16.8161 & -1.4679 & -20.1029 & -1.2207 & 1.5418 \\ 51.0744 & 486.7451 & 13.0354 & 101.0287 & 8.6774 & 8.6872 & 9.1953 & 52.8509 \\ -0.7887 & 13.0354 & 9.1765 & 17.2759 & 0.8705 & -0.8746 & 1.4118 & -4.9874 \\ 16.8161 & 101.0287 & 17.2759 & 244.3918 & 4.3396 & -39.0897 & 6.6145 & 46.1743 \\ -1.4679 & 8.6774 & 0.8705 & 4.3396 & 8.7297 & 8.4651 & -0.8661 & 4.9237 \\ -20.1029 & 8.6872 & -0.8746 & -39.0897 & 8.4651 & 239.4326 & 5.1983 & 13.5890 \\ -1.2207 & 9.1953 & 1.4118 & 6.6145 & -0.8661 & 5.1983 & -0.9111 & -0.3332 \\ 1.5418 & 52.8509 & -4.9874 & 46.1743 & 4.9237 & 13.5890 & -0.3332 & 25.5901 \end{bmatrix}$$

$$D_{P,1} = \begin{bmatrix} 9.5250 & 6.9870 & -1.0628 & -2.8941 \\ 6.9870 & 65.6791 & 0.0831 & -3.5221 \\ -1.0628 & 0.0831 & 9.9204 & 3.9213 \\ -2.8941 & -3.5221 & 3.9213 & 69.1830 \end{bmatrix}, \quad D_{P,2} = \begin{bmatrix} 9.4393 & 9.0572 & -1.0995 & 1.3359 \\ 9.0572 & 69.1088 & 2.2309 & 2.5806 \\ -1.0995 & 2.2309 & 9.7363 & 6.1760 \\ 1.3359 & 2.5806 & 6.1760 & 71.0502 \end{bmatrix}.$$

To save space, the numerical data of  $D_{R,1}$  and  $D_{R,2}$  is not listed here. Since LMI (208) and (209) are solvable for  $A_0, A_1, A_2$ , it can be concluded that  $A(\rho)$  is Hurwitz for all  $\rho \in [-1, +1] \times [-1, +1]$ .

We now give an example as follows, in which  $A(\rho)$  is not Hurwitz for all  $\rho \in [-1, +1] \times [-1, +1]$ .

**Example 6.13** Consider the two-parameter dependent matrix  $A(\rho)$  as in Example 6.6, where

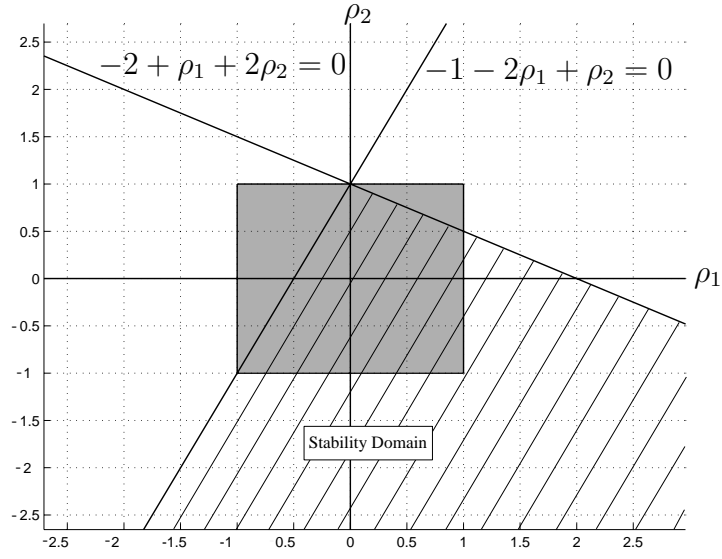
$$A(\rho) = \begin{bmatrix} -2 & 7 \\ 0 & -1 \end{bmatrix} + \rho_1 \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} + \rho_2 \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$$

The stability domain for this  $A(\rho)$  is described as the following inequalities.

$$\lambda_1(A(\rho)) = -2 + \rho_1 + 2\rho_2 < 0$$

$$\lambda_2(A(\rho)) = -1 - 2\rho_1 + \rho_2 < 0$$

As Fig. 11 shows, the set  $[-1, +1] \times [-1, +1]$  in  $\rho_1 - \rho_2$  space is not a subset of the stability domain where  $A(\rho)$  is Hurwitz and both of the two eigenvalue inequalities are satisfied. We



**Figure 11:** Whole Stability Domain

now apply Theorem 6.4 to test whether  $A(\rho)$  is Hurwitz for all  $\rho \in [-1, +1] \times [-1, +1]$ .

For this specific problem,

$$n = 2, \quad m = 2, \quad \bar{k} = \frac{1}{2}n(n+1) - 1 = 2, \quad \bar{\alpha} = \lceil \frac{\bar{k}}{2} \rceil + 1 = 2,$$

$$m_1 = (\bar{\alpha} - 1)^m n = 2, \quad m_2 = \bar{\alpha}^m n = 8$$

$$K_1 = [\bar{\alpha}, \bar{\alpha}]^T = [2, 2]^T, \quad K_2 = [(\bar{\alpha} + 1), (\bar{\alpha} + 1)]^T = [3, 3]^T,$$

$$P_\Sigma \in \mathbb{R}^{8 \times 8}, \quad D_{P,1}, D_{P,2} \in \mathbb{R}^{4 \times 4}, \quad D_{R,1}, D_{R,2} \in \mathbb{R}^{16 \times 16}$$

and

$$\mathcal{C}_{K_1} = \begin{bmatrix} \bar{C}_{K_1,1,1} \\ \bar{C}_{K_1,1,2} \\ \bar{C}_{K_1,2,1} \\ \bar{C}_{K_1,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix}, \quad \mathcal{J}_{K_1} = \begin{bmatrix} \bar{J}_{K_1,1,1} \\ \bar{J}_{K_1,1,2} \\ \bar{J}_{K_1,2,1} \\ \bar{J}_{K_1,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix},$$

$$\mathcal{C}_{K_2} = \begin{bmatrix} \bar{C}_{K_2,1,1} \\ \bar{C}_{K_2,1,2} \\ \bar{C}_{K_2,2,1} \\ \bar{C}_{K_2,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \hat{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \hat{J}_2 \otimes \check{J}_2 \otimes I_2 \end{bmatrix}, \quad \mathcal{J}_{K_2} = \begin{bmatrix} \bar{J}_{K_2,1,1} \\ \bar{J}_{K_2,1,2} \\ \bar{J}_{K_2,2,1} \\ \bar{J}_{K_2,2,2} \end{bmatrix} = \begin{bmatrix} \hat{J}_2 \otimes \check{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \check{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \check{J}_2 \otimes I_2 \end{bmatrix},$$

and

$$\begin{aligned} H_{\Sigma} &= \hat{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ F_{\Sigma} &= \hat{J}_2 \otimes \hat{J}_2 \otimes A_0 + \hat{J}_2 \otimes \check{J}_2 \otimes A_1 + \check{J}_2 \otimes \hat{J}_2 \otimes A_2 . \end{aligned}$$

Since LMI (208) and (209) are not feasible simultaneously and according to Theorem 6.4,  $A(\rho)$  is not Hurwitz for all  $\rho \in [-1, +1] \times [-1, +1]$ . The direct eigenvalue analysis and Fig. 11 agree with the result by Theorem 6.4.

## 6.4 Conclusions

The results in this chapter generalize those of Chapter 4. It is shown that the stability of multi-parameter LTIPD systems of the form  $\dot{x}(t) = A(\rho)x(t)$  where  $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2 + \dots + \rho_m A_m$ , is equivalent to the existence of a Lyapunov function  $P(\rho)$ . In this chapter it is proved that a multi-parameter dependent Lyapunov function  $P(\rho)$  of polynomial type of given degree exists if and only if the matrix  $A(\rho)$  is Hurwitz. The degree of the polynomial dependence of  $P(\rho)$  on  $\rho$  can be reduced significantly in case the matrices  $A_i$ ,  $i = 1, 2, \dots, m$  are rank deficient. When the parameter vector  $\rho$  is inside a compact set  $[-1, +1]^m$ , the two Lyapunov matrix inequalities  $P(\rho) > 0$  and  $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$  that characterize stability can be cast into two nonconservative LMIs, which can be used to test the stability of multi-parameter affinely dependent LTI systems.

## CHAPTER VII

# STABILITY ANALYSIS OF LPV TIME-DELAYED SYSTEMS

The need for stability analysis of metal cutting process, which can be described as an LPV time-delay system, motivated the research work of this chapter. In addition to the metal cutting process, several linear time-delayed systems [82, 23, 81] depend on parameters whose values are time-varying but not known a priori. Assuming that the parameters enter the system dynamics *without delay*, an LPV time-delayed system has the form

$$\dot{x}(t) = A(\gamma(t))x(t) + A_d(\gamma(t))x(t - \tau) \quad (210)$$

In (210)  $\tau$  is a constant, unknown delay with  $\tau \in [0, \bar{\tau}]$  and  $\gamma$  is a parameter vector that is assumed to belong to a known polytope  $\Gamma$ . Often, it is also known that the rate of  $\gamma$  belongs to a given polytope,  $\Gamma_r$ . Typically, one is interested in deriving conditions that will guarantee stability for system (210) for all  $(\gamma, \dot{\gamma}) \in \Gamma \times \Gamma_r$ , and all  $\tau \in [0, \bar{\tau}]$ . In cases where there are no restrictions in the variation rate of  $\gamma$  we have  $\Gamma_r = \mathbb{R}$ . In addition, if the stability conditions hold for all  $\tau \in [0, \bar{\tau}]$  with  $\bar{\tau} < \infty$  then the stability is referred to as *delay-dependent* stability. If the conditions hold for all  $\tau \in [0, \infty)$ , then the stability is referred to as *delay-independent* stability, since stability is ensured for any amount of delay.

Stability analysis and synthesis results have been reported for LPV systems and LTI time-delay systems [11, 4, 83, 9, 30, 49, 31, 79]. However, the stability theory of LPV, *time-delayed* systems is less developed. The specific contribution of the research work in this chapter is to derive both delay-independent and delay-dependent stability conditions for LPV, *time-delayed* systems. These stability conditions are expressed in terms of LMIs and are thus computationally tractable. We restrict our discussion to the scalar parameter LPV case. Our results can be generalized to the case when the parameter  $\gamma$  is a vector, but the

derivations become more cumbersome. Some of our results in this chapter has already been publish in [97, 96].

Linear Parameter Varying (LPV) systems can be considered as a special class of Linear Time-Varying (LTV) systems. The main difference with LTV systems is that in LPV systems the time-dependence of the system matrices  $A$  and  $A_d$  in (210) is not known a priori but is given only implicitly by the parameter  $\gamma(t)$  which is assumed to be a priori unknown.

The following stability condition for LTV (time-delayed) systems [45, 76] provides the main tool used to show (global) asymptotic stability in this chapter.

**Theorem 7.1 ([58, 76])** *Given some  $\tau > 0$ , assume there exists a positive definite, continuous functional  $V : \mathbb{R}_+ \times \mathcal{C}_\tau \rightarrow \mathbb{R}_+$ , with infinitesimal upper bound whose derivative  $\dot{V}$  is a negative definite functional. Then the trivial solution of the LTV, time-delayed system  $\dot{x}(t) = A(t)x(t) + A_d(t)x(t - \tau)$  is (globally) uniformly asymptotically stable.*

## 7.1 Delay-Independent Stability

The results in this section deal with systems where the delay  $\tau$  is unbounded that is,  $\tau \in [0, \infty)$ . Since stability is ensured for every positive delay  $\tau$ , the stability conditions are delay-independent. Several stability tests are derived in the sequel.

In the following, the dependence on the time  $t$  has been suppressed for notational simplicity. From now on, it will be tacitly assumed that all parameter-varying matrices depend continuously on the parameter  $\gamma$ .

**Theorem 7.2** *Consider the LPV time-delayed system (210) and let  $\gamma \in \Gamma = [\underline{\gamma}, \bar{\gamma}]$ . Consider a constant matrix  $P$  and a matrix-valued function  $Q : \Gamma \rightarrow \mathbb{R}^{n \times n}$  such that*

$$P > 0, \quad Q(\gamma) > 0, \quad \forall \gamma \in \Gamma \quad (211)$$

and

$$M_1(\gamma_1, \gamma_2) = \begin{bmatrix} PA(\gamma_1) + A^T(\gamma_1)P + Q(\gamma_1) & PA_d(\gamma_1) \\ A_d^T(\gamma_1)P & -Q(\gamma_2) \end{bmatrix} < 0 \quad (212)$$

for all  $\gamma_i \in \Gamma$ ,  $i = 1, 2$ . Then the system (210) is asymptotically stable for all  $\gamma \in \Gamma$  and  $\tau \in [0, \infty)$ .

*Proof.* Consider the following Lyapunov-Krasovskii functional  $V : \mathbb{R}_+ \times \mathcal{C}_\tau \rightarrow \mathbb{R}_+$

$$V(t, x_t) = x^T(t)Px(t) + \int_{-\tau}^0 x^T(t+\theta)Q(\gamma(t+\theta))x(t+\theta) d\theta$$

where  $P$  and  $Q(\gamma)$  as in (211), and  $x_t \in \mathcal{C}_\tau$ ,  $x_t(0) = x(t)$ ,  $x_t(-\tau) = x(t-\tau)$ . From (211) and Lemma 2.6, it follows that  $V$  is positive definite with an infinitesimal upper bound. The derivative of  $V$  along the trajectories of (210) is

$$\begin{aligned} \dot{V}(t, x_t) &= 2x^T(t)PA(\gamma(t))x(t) + 2x^T(t)PA_d(\gamma(t))x(t-\tau) \\ &\quad + x^T(t)Q(\gamma(t))x(t) - x^T(t-\tau)Q(\gamma(t-\tau))x(t-\tau) \end{aligned}$$

or

$$\dot{V}(t, x_t) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T M_1(\gamma_1, \gamma_2) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \quad (213)$$

where  $\gamma_1 = \gamma(t)$  and  $\gamma_2 = \gamma(t-\tau)$ . Inequality (212) implies that the matrix  $M_1(\gamma_1, \gamma_2)$  is negative definite for all  $\gamma_1, \gamma_2 \in \Gamma$ . Since  $\Gamma$  is compact, then

$$-\dot{V}(t, x_t) > -\min_{\gamma_1, \gamma_2} \lambda_{\max}[M_1(\gamma_1, \gamma_2)] (|x(t)|^2 + |x(t-\tau)|^2) > c |x(t)|^2$$

where  $c = -\min_{\gamma_1, \gamma_2} \lambda_{\max}[M_1(\gamma_1, \gamma_2)] > 0$  and system (210) is asymptotically stable.  $\blacksquare$

Conditions (211)-(212) represent an infinite-dimensional set of LMI's. Gridding (see Section 7.1.2) can be used to project on a finite set of LMI's. In case the system matrices have a polynomial dependence on the parameter  $\gamma$ , the following result may be useful.

**Theorem 7.3** Consider the LPV time-delayed system (210) and assume that

$$A(\gamma) = A_0 + \gamma A_1 + \gamma^2 A_2 \quad \text{and} \quad A_d(\gamma) = A_{d0} + \gamma A_{d1}$$

where  $\gamma \in \Gamma$ , with  $\Gamma$  any compact sub-interval of  $\mathbb{R}$ . If there exist constant, positive-definite matrices  $P$  and  $Q$  such that

$$M_2 = \begin{bmatrix} A_0^T P + P A_0 + Q & P A_{d0} & P A_1 \\ A_{d0}^T P & -Q & A_{d1}^T P \\ A_1^T P & P A_{d1} & A_2^T P + P A_2 \end{bmatrix} < 0$$

then system (210) is asymptotically stable for any value of the parameter  $\gamma \in \Gamma$  and any  $\tau \in [0, \infty)$ .

*Proof.* Consider the following Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t) P x(t) + \int_{-\tau}^0 x^T(t + \theta) Q x(t + \theta) d\theta$$

From Lemma 2.6,  $V$  is positive definite and has an infinitesimal upper bound. The derivative of  $V$  along the trajectories of (210) is

$$\begin{aligned} \dot{V} &= 2x^T(t) P (A_0 + \gamma A_1 + \gamma^2 A_2) x(t) + x^T(t) Q x(t) \\ &\quad + 2x^T(t) P (A_{d0} + \gamma A_{d1}) x(t - \tau) - x^T(t - \tau) Q x(t - \tau) \end{aligned}$$

The last equation can be written as

$$\dot{V}(t) = \begin{bmatrix} x(t) \\ x(t - \tau) \\ \gamma x(t) \end{bmatrix}^T M_2 \begin{bmatrix} x(t) \\ x(t - \tau) \\ \gamma x(t) \end{bmatrix} \quad (214)$$

Since  $\Gamma$  is compact, the previous inequality holds uniformly for all  $\gamma \in \Gamma$ . Hence  $\dot{V}$  is negative definite and from Theorem 7.1 the system (210) is asymptotically stable [45]. ■

**Remark 7.1** In Theorem 7.3 the set  $\Gamma$  can be arbitrarily large. Hence, the conditions of the theorem guarantee that system (210) is stable for any (bounded) values of the parameter

$\gamma \in \mathbb{R}$ . It requires, however, that  $A_2^T P + P A_2 < 0$ , and  $A_0^T P + P A_0 + Q < 0$ , i.e., the matrices  $A_0$  and  $A_2$  must be Hurwitz. This condition induces unnecessary conservatism. Assuming that the parameter  $\gamma$  is known to belong to a *known* compact interval, Theorem 7.2 can be used to relax the conditions for delay-independent stability for (210).

### 7.1.1 Stability under Bounded Parameter Variation

Theorems 7.2 and 7.3 did not consider the time variation of the parameter  $\gamma$ . In that respect, Theorems 7.2 and 7.3 can be potentially conservative, since they ensure – in principle – stability for arbitrarily fast variations of  $\gamma$ . In particular, in Theorem 7.2  $\gamma(t)$  and  $\gamma(t-\tau)$  are treated as independent variables. Nonetheless, this does not induce any extra conservatism even for the case when  $\dot{\gamma}$  is bounded by a (relatively) small upper bound. Unless  $\dot{\gamma} = 0$  then  $\gamma(t)$  and  $\gamma(t-\tau)$  must be treated as independent since the delay may be arbitrarily large. Hence, for truly delay-independent results, the bound of  $\dot{\gamma}$  does not impose any constraints between  $\gamma(t)$  and  $\gamma(t-\tau)$ . If, on the other hand, the delay is known to belong to a bounded interval, then the treatment of  $\gamma(t)$  and  $\gamma(t-\tau)$  as independent may cause extra conservatism, especially for small bounds on the parameter variation rate. Similarly, if  $\gamma$  varies very fast then it is expected that (for delay-independent stability)  $\gamma(t)$  and  $\gamma(t-\tau)$  can still be treated independently, even for small values of the delay. See also the discussion at the end of Section 7.1.

Next, stability tests are derived that take explicitly into account the knowledge of the bound of the rate of variation of the parameter.

**Theorem 7.4** *Consider the LPV time-delayed system (210) with  $\gamma \in \Gamma = [\underline{\gamma}, \bar{\gamma}]$  and  $\dot{\gamma} \in \Gamma_r = [\underline{\dot{\gamma}}, \bar{\dot{\gamma}}]$ . Consider the matrix valued functions  $P : \Gamma \rightarrow \mathbb{R}^{n \times n}$  and  $Q : \Gamma \rightarrow \mathbb{R}^{n \times n}$  such that*

$$P(\gamma) > 0, \quad Q(\gamma) > 0, \quad \forall \gamma \in \Gamma \quad (215)$$



and

$$M_3(\gamma_1, \gamma_2, \nu) = \begin{bmatrix} P(\gamma_1)A(\gamma_1) + ( )^T + Q(\gamma_1) + \frac{\partial P}{\partial \gamma} \nu & P(\gamma_1)A_d(\gamma_1) \\ A_d^T(\gamma_1)P(\gamma_1) & -Q(\gamma_2) \end{bmatrix} < 0 \quad (216)$$

for all  $\gamma_1, \gamma_2 \in \Gamma$  and  $\nu \in \Gamma_r$ . Then the system (210) is asymptotically stable for all  $(\gamma, \dot{\gamma}) \in \Gamma \times \Gamma_r$  and  $\tau \in [0, \infty)$ .

*Proof.* Consider the following Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t)P(\gamma(t))x(t) + \int_{-\tau}^0 x^T(t+\theta)Q(\gamma(t+\theta))x(t+\theta) d\theta \quad (217)$$

From (215),  $V$  is positive definite with an infinitesimal upper bound. The derivative of  $V$  along the trajectories of (210) is

$$\begin{aligned} \dot{V}(t, x_t) &= 2x^T(t)P(\gamma(t))A(\gamma(t))x(t) + x^T(t)\frac{\partial P}{\partial \gamma}\dot{\gamma}x(t) \\ &+ 2x^T(t)P(\gamma(t))A_d(\gamma(t))x(t-\tau) \\ &+ x^T(t)Q(\gamma(t))x(t) - x^T(t-\tau)Q(\gamma(t-\tau))x(t-\tau) \end{aligned}$$

or

$$\dot{V}(t, x_t) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T M_3(\gamma(t), \gamma(t-\tau), \dot{\gamma}(t)) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}$$

Inequality (216) implies that  $M_3$  is negative definite for all  $\gamma \in \Gamma$  and  $\dot{\gamma} \in \Gamma_r$ . Since  $\Gamma$  and  $\Gamma_r$  are compact,  $-\dot{V}(t, x_t) > -c(|x(t)|^2 + |x(t-\tau)|^2) > c|x(t)|^2$  where  $c = -\min_{\gamma_1, \gamma_2, \nu} \lambda_{\max}[M_3(\gamma_1, \gamma_2, \nu)] > 0$  and thus, the system (210) is asymptotically stable. ■

### 7.1.2 Gridding the Parameter Space

Eqs. (215)-(216) or (211)-(212) represent an infinite dimensional system of LMI's. A common way to reduce these conditions to a finite set of LMI's is to use gridding of the parameter space. According to this approach, one selects a set of basis functions  $f_i(\gamma)$ , ( $i =$

$1, 2, \dots, n_1)$  and  $g_i(\gamma)$ , ( $i = 1, 2, \dots, n_2$ ) and expands  $P$  and  $Q$  in terms of these basis functions as

$$P(\gamma) = \sum_{i=1}^{n_1} P_i f_i(\gamma) \quad \text{and} \quad Q(\gamma) = \sum_{j=1}^{n_2} Q_j g_j(\gamma) \quad (218)$$

One then seeks matrices  $P_i$ , ( $i = 1, 2, \dots, n_1$ ) and  $Q_j$ , ( $j = 1, 2, \dots, n_2$ ) such that  $\sum_{i=1}^{n_1} P_i f_i(\gamma) > 0$  and  $\sum_{j=1}^{n_2} Q_j g_j(\gamma) > 0$  for all  $\gamma \in \Gamma$  and

$$\begin{bmatrix} \begin{pmatrix} (\sum_{i=1}^{n_1} P_i f_i(\gamma_1)) A(\gamma_1) + \\ A^T(\gamma_1) (\sum_{i=1}^{n_1} P_i f_i(\gamma_1)) + \\ \sum_{i=1}^{n_1} P_i \frac{\partial f_i}{\partial \gamma} \nu + \sum_{j=1}^{n_2} Q_j g_j(\gamma_1) \end{pmatrix} & (\sum_{i=1}^{n_1} P_i f_i(\gamma_1)) A_d(\gamma_1) \\ A_d^T(\gamma_1) (\sum_{i=1}^{n_1} P_i f_i(\gamma_1)) & - \sum_{j=1}^{n_2} Q_j g_j(\gamma_2) \end{pmatrix} < 0 \quad (219)$$

for all  $\gamma_i \in \Gamma$ ,  $i = 1, 2$ . The solution of (218)-(219), for instance, is searched over a finite number (grid) of the parameter values. After a solution is found, it is typically validated by testing it on a finer grid. Gridding leads to computationally expensive stability tests. It can be used when the dimension of the parameter vector  $\gamma$  is low. If the number of parameters is large, gridding can be computationally prohibitive, since the number of LMI's to be solved increases exponentially with the number of parameters. In order to get computationally tractable tests, we next assume that the system matrices  $A(\gamma)$  and  $A_d(\gamma)$  in (210) have a specific polynomial dependence on the parameter  $\gamma$ . Our results will also hold for more complex (non-polynomial) parameter dependencies as long as a polynomial approximation of the parameter dependence holds within the parameter range of interest.

### 7.1.3 A Relaxation Approach

The results of Theorem 7.2 and 7.4 require gridding of the parameter spaces  $\Gamma$  and  $\Gamma \times \Gamma_r$ , respectively. This can be cumbersome since for fine gridding, many matrix inequalities have to be solved simultaneously. In certain cases, the parameter dependence on the matrices  $A$  and  $A_d$  is relatively simple (low order polynomial) and gridding may be avoided using multi-convexity arguments and relaxation methods at the expense of increasing conservatism [27,

74]. Next, several special cases are explored when gridding can be avoided. In order to prove the main results of this section, the following two lemmas will be used in the sequel.

**Lemma 7.1** *Consider the following parameter dependent matrix  $F(\gamma) = \gamma^2 F_2 + \gamma F_1 + F_0$  where  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ . If  $F_2 \geq 0$ , then  $F(\gamma)$  is a convex, matrix-valued function, that is,*

$$\lambda F(\gamma_1) + (1 - \lambda)F(\gamma_2) \geq F(\lambda\gamma_1 + (1 - \lambda)\gamma_2), \quad \forall \gamma_1, \gamma_2 \in [\underline{\gamma}, \bar{\gamma}] \quad (220)$$

*for any scalar  $0 \leq \lambda \leq 1$ . If  $F_2 > 0$  then  $F(\gamma)$  is a strictly convex, matrix-valued function, i.e., (220) is satisfied with strict inequality for all  $0 < \lambda < 1$ . Moreover, if  $F_2 \geq 0$  and  $F(\gamma^\#) < 0$  for  $\gamma^\# \in \{\underline{\gamma}, \bar{\gamma}\}$ , then  $F(\gamma) < 0$  for all  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ .*

*Proof.*

$$\begin{aligned} \lambda F(\gamma_1) + (1 - \lambda)F(\gamma_2) &= \lambda(\gamma_1^2 F_2 + \gamma_1 F_1 + F_0) + (1 - \lambda)(\gamma_2^2 F_2 + \gamma_2 F_1 + F_0) \\ &= (\lambda\gamma_1^2 + (1 - \lambda)\gamma_2^2)F_2 + (\lambda\gamma_1 + (1 - \lambda)\gamma_2)F_1 + F_0 \end{aligned}$$

and

$$F(\lambda\gamma_1 + (1 - \lambda)\gamma_2) = (\lambda\gamma_1 + (1 - \lambda)\gamma_2)^2 F_2 + (\lambda\gamma_1 + (1 - \lambda)\gamma_2)F_1 + F_0$$

Since the function  $f(x) = x^2$  is convex,  $\lambda\gamma_1^2 + (1 - \lambda)\gamma_2^2 > (\lambda\gamma_1 + (1 - \lambda)\gamma_2)^2$  for any  $0 < \lambda < 1$ . Together with  $F_2 \geq 0$ , one has

$$(\lambda\gamma_1^2 + (1 - \lambda)\gamma_2^2)F_2 \geq (\lambda\gamma_1 + (1 - \lambda)\gamma_2)^2 F_2$$

and thus,

$$\begin{aligned} &(\lambda\gamma_1^2 + (1 - \lambda)\gamma_2^2)F_2 + (\lambda\gamma_1 + (1 - \lambda)\gamma_2)F_1 + F_0 \\ &\geq (\lambda\gamma_1 + (1 - \lambda)\gamma_2)^2 F_2 + (\lambda\gamma_1 + (1 - \lambda)\gamma_2)F_1 + F_0 \end{aligned}$$

Therefore, Inequality (220) holds. The proof is complete. ■

**Lemma 7.2** *Consider the following parameter dependent matrix*

$$F(\gamma_1, \gamma_2, \gamma_3) = \gamma_1^2 F_2 + \gamma_1 F_1 + \gamma_2^2 F_3 + \gamma_2 F_4 + F_0(\gamma_3), \quad \text{where} \quad F_0(\gamma_3) = F_{01} + \gamma_3 F_{02}$$

where  $\gamma_i \in [\underline{\gamma}_i, \bar{\gamma}_i] = \Gamma_i$  for  $i = 1, 2, 3$ . Let  $\Gamma_i^\# = \{\underline{\gamma}_i, \bar{\gamma}_i\}$  denote the vertices of  $\Gamma_i$  for  $i = 1, 2, 3$ . If  $F_3 \geq 0, F_2 \geq 0$  and  $F(\gamma_1^\#, \gamma_2^\#, \gamma_3^\#) < 0$  for  $(\gamma_1^\#, \gamma_2^\#, \gamma_3^\#) \in \Gamma_1^\# \times \Gamma_2^\# \times \Gamma_3^\#$  then  $F(\gamma_1, \gamma_2, \gamma_3) < 0$  for all  $(\gamma_1, \gamma_2, \gamma_3) \in \Gamma_1 \times \Gamma_2 \times \Gamma_3$ .

*Proof.* See [27]. ■

In the following, it is assumed that  $\Gamma = [-1, 1]$ . In case  $\Gamma \neq [-1, 1]$ , one may choose  $\tilde{\gamma} = [2\gamma - (\bar{\gamma} + \underline{\gamma})]/(\bar{\gamma} - \underline{\gamma})$ , such that  $\tilde{\gamma} \in [-1, 1]$ . This simplification can always be made without loss of generality and results in more compact formulas.

**Theorem 7.5** *Consider the system (210) where*

$$A(\gamma) = A_0 + \gamma A_1 + \gamma^2 A_2 \quad \text{and} \quad A_d(\gamma) = A_{d0} + \gamma A_{d1} + \gamma^2 A_{d2} \quad (221)$$

where  $\gamma \in [-1, 1]$ , and  $\dot{\gamma} \in [\underline{\dot{\gamma}}, \bar{\dot{\gamma}}]$ . Assume that there exist negative semi-definite matrices  $Q_4, Q_2, P_2$ , positive-definite matrices  $Q_0, P_0$  and symmetric matrices  $Q_1, Q_3, P_1$  such that

$$Q_0 \pm Q_1 + 2Q_2 > 0, \quad -Q_2 \pm Q_3 + Q_4 \geq 0, \quad (222)$$

$$P_0 \pm P_1 + P_2 > 0 \quad (223)$$

$$N_2 + \gamma_1^\# N_1 + N_3 + \gamma_2^\# N_4 + N_0(\nu) < 0 \quad (224)$$

where  $\gamma_i^\# \in \{-1, 1\}$  and  $\nu \in \{\underline{\dot{\gamma}}, \bar{\dot{\gamma}}\}$ , and

$$N_2 = \alpha_1 \Theta_1 + \alpha_2 \Theta_2 + \Theta_3 \geq 0, \quad N_1 = \frac{1-\alpha_1}{2} \Theta_1 + \frac{3-3\alpha_2}{4} \Theta_2 + \Theta_4 \quad (225)$$

$$N_0(\dot{\gamma}) = \frac{3\alpha_1-3}{16} \Theta_1 + \frac{\alpha_2-1}{4} \Theta_2 + \Theta_5(\dot{\gamma}) \quad (226)$$

$$N_3 = \begin{bmatrix} 0 & 0 \\ 0 & -Q_4 - (1-\beta)Q_3 - Q_2 \end{bmatrix} \geq 0, \quad N_4 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{3}{4}\beta Q_3 - Q_1 \end{bmatrix} \quad (227)$$

where

$$\begin{aligned}
\Theta_1 &= \begin{bmatrix} A_2^T P_2 + P_2 A_2 + Q_4 & P_2 A_{d2} \\ A_{d2}^T P_2 & 0 \end{bmatrix} \\
\Theta_2 &= \begin{bmatrix} \begin{pmatrix} A_1^T P_2 + A_2^T P_1 \\ + P_1 A_2 + P_2 A_1 + Q_3 \end{pmatrix} & P_1 A_{d2} + P_2 A_{d1} \\ A_{d2}^T P_1 + A_{d1}^T P_2 & 0 \end{bmatrix} \\
\Theta_3 &= \begin{bmatrix} \begin{pmatrix} A_2^T P_0 + A_0^T P_2 \\ + A_1^T P_1 + ( )^T + Q_2 \end{pmatrix} & \begin{pmatrix} P_2 A_{d0} + P_1 A_{d1} \end{pmatrix} \\ A_{d1}^T P_1 + A_{d0}^T P_2 & A_{d2}^T P_0 A_{d2} \end{bmatrix} \\
\Theta_4 &= \begin{bmatrix} \begin{pmatrix} A_1^T P_0 + P_0 A_1 \\ + A_0^T P_1 + P_1 A_0 + Q_1 \end{pmatrix} & P_0 A_{d1} + P_1 A_{d0} \\ A_{d1}^T P_0 + A_{d0}^T P_1 & 0 \end{bmatrix} \\
\Theta_5(\dot{\gamma}) &= \begin{bmatrix} \begin{pmatrix} A_0^T P_0 + P_0 A_0 \\ -2\nu_m P_2 + \dot{\gamma} P_1 + Q_0 \end{pmatrix} & P_0 A_{d0} \\ A_{d0}^T P_0 & -Q_0 + \frac{1}{4}\beta Q_3 \end{bmatrix}
\end{aligned}$$

where  $\nu_m = \max\{|\underline{\dot{\gamma}}|, |\bar{\dot{\gamma}}|\}$ , and where the pair  $(\alpha_1, \alpha_2)$  takes any of the four possible combinations  $(\alpha_1, \alpha_2) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$  and  $\beta \in \{0, 1\}$ . Then the system (210) is asymptotically stable for all  $\gamma \in [-1, 1]$ , all  $\dot{\gamma} \in [\underline{\dot{\gamma}}, \bar{\dot{\gamma}}]$  and all  $\tau \in [0, \infty)$ .

*Proof.* Consider the Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t)P(\gamma(t))x(t) + \int_{-\tau}^0 x^T(t+\theta)Q(\gamma(t+\theta))x(t+\theta) d\theta$$

where

$$P(\gamma) = P_0 + \gamma P_1 + \gamma^2 P_2 \quad \text{and} \quad Q(\gamma) = Q_0 + \gamma Q_1 + \gamma^2 Q_2 + \gamma^3 Q_3 + \gamma^4 Q_4 \quad (228)$$

Eq. (223) implies that  $P(\gamma^\#) > 0$  for  $\gamma^\# \in \{-1, 1\}$ . Since  $P_2 \leq 0$ ,  $-P(\gamma)$  is convex. From Lemma 7.1 it follows that  $P(\gamma) > 0$  uniformly, for all  $\gamma \in [-1, 1]$ . Now write  $Q(\gamma)$  as follows

$$Q(\gamma) = (2\gamma^2 Q_2 + \gamma Q_1 + Q_0) + \gamma^2(\gamma^2 Q_4 + \gamma Q_3 - Q_2)$$

Since  $Q_2 \leq 0$  and  $Q_4 \leq 0$ , then (222) along with Lemma 7.1 imply that  $Q(\gamma) > 0$  uniformly for all  $\gamma \in [-1, 1]$ . It follows that  $V$  is positive definite with an infinitesimal upper bound. The derivative of  $V$  along the system (210) is

$$\begin{aligned}\dot{V}(t) = & 2x^T(t)(P_0 + \gamma_1 P_1 + \gamma_1^2 P_2)(A_0 + \gamma_1 A_1 + \gamma_1^2 A_2)x(t) \\ & + x^T(t)(\dot{\gamma}_1 P_1 + 2\gamma_1 \dot{\gamma}_1 P_2)x(t) \\ & + 2x^T(t)(P_0 + \gamma_1 P_1 + \gamma_1^2 P_2)(A_{d0} + \gamma_1 A_{d1} + \gamma_1^2 A_{d2})x(t - \tau) \\ & + x^T(t)(Q_0 + \gamma_1 Q_1 + \gamma_1^2 Q_2 + \gamma_1^3 Q_3 + \gamma_1^4 Q_4)x(t) \\ & - x^T(t - \tau)(Q_0 + \gamma_2 Q_1 + \gamma_2^2 Q_2 + \gamma_2^3 Q_3 + \gamma_2^4 Q_4)x(t - \tau)\end{aligned}$$

where  $\gamma_1 = \gamma(t)$ ,  $\gamma_2 = \gamma(t - \tau)$ . Since  $P_2 \leq 0$  and  $2\gamma_1 \dot{\gamma}_1 x^T(t)P_2 x(t) \leq -2\nu_m x^T(t)P_2 x(t)$ , where  $\nu_m = \max\{|\dot{\gamma}|, |\bar{\gamma}|\}$ , one can then rewrite the equation for  $\dot{V}$  as

$$\begin{aligned}\dot{V}(t) \leq & \gamma_1^4 \left\{ 2x^T(t)(P_2 A_2 + 0.5Q_4)x(t) + 2x^T(t)P_2 A_{d2}x(t - \tau) \right\} \\ & - \gamma_2^4 x^T(t - \tau)Q_4 x(t - \tau) - \gamma_2^3 x^T(t - \tau)Q_3 x(t - \tau) \\ & + \gamma_1^3 \left\{ 2x^T(t)(P_1 A_2 + P_2 A_1 + 0.5Q_3)x(t) + 2x^T(t)(P_1 A_{d2} + P_2 A_{d1})x(t - \tau) \right\} \\ & + \gamma_1^2 \left\{ 2x^T(t)(P_0 A_2 + P_1 A_1 + P_2 A_0 + 0.5Q_2)x(t) \right. \\ & \quad \left. + 2x^T(t)(P_0 A_{d2} + P_1 A_{d1} + P_2 A_{d0})x(t - \tau) \right\} - \gamma_2^2 x^T(t - \tau)Q_2 x(t - \tau) \\ & + \gamma_1 \left\{ 2x^T(t)(P_0 A_1 + P_1 A_0 + 0.5Q_1)x(t) + 2x^T(t)(P_0 A_{d1} + P_1 A_{d0})x(t - \tau) \right\} \\ & - \gamma_2 x^T(t - \tau)Q_1 x(t - \tau) - x^T(t - \tau)Q_0 x(t - \tau) \\ & + 2x^T(t)(P_0 A_0 - \nu_m P_2 + 0.5Q_0 + 0.5\dot{\gamma}P_1)x(t) + 2x^T(t)P_0 A_{d0}x(t - \tau)\end{aligned}\tag{229}$$

Notice now that since  $P_0 > 0$  it follows that the inequality

$$2x^T(t)P_0 A_{d2}x(t - \tau) \leq x^T(t)P_0 x(t) + x^T(t - \tau)A_{d2}^T P_0 A_{d2}x(t - \tau)$$

holds. Also, it can be immediately verified that for all  $\gamma \in [-1, 1]$ <sup>1</sup> the following inequalities hold

$$\gamma^2 \geq \gamma^4 \geq \frac{1}{2}\gamma - \frac{3}{16}, \quad \gamma^2 \geq \gamma^3 \geq \frac{3}{4}\gamma - \frac{1}{4}\tag{230}$$

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<sup>1</sup>See, for example, [74].

Then  $\gamma^4 y \leq \gamma^2 y$  if  $y \geq 0$  and  $\gamma^4 y \leq (\frac{1}{2}\gamma - \frac{3}{16})y$  if  $y < 0$ . Therefore,

$$\gamma^4 y \leq \max \left\{ \gamma^2 y, \left( \frac{1}{2}\gamma - \frac{3}{16} \right) y \right\} \quad (231)$$

and

$$\gamma^3 y \leq \max \left\{ \gamma^2 y, \left( \frac{3}{4}\gamma - \frac{1}{4} \right) y \right\} \quad (232)$$

These inequalities imply that

$$\begin{aligned} & \gamma_1^4 \left\{ 2x^T(t) (P_2 A_2 + 0.5Q_4) x(t) + 2x^T(t) P_2 A_{d2} x(t - \tau) \right\} \\ & \leq \max \left\{ \left( \frac{\gamma_1}{2} - \frac{3}{16} \right) \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}^T \Theta_1 \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}, \right. \\ & \quad \left. \gamma_1^2 \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}^T \Theta_1 \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix} \right\} \end{aligned} \quad (233)$$

and

$$\begin{aligned} & \gamma_1^3 \left\{ 2x^T(t) (P_1 A_2 + P_2 A_1 + 0.5Q_3) x(t) + 2x^T(t) (P_1 A_{d2} + P_2 A_{d1}) x(t - \tau) \right\} \\ & \leq \max \left\{ \gamma_1^2 \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}^T \Theta_2 \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}, \left( \frac{3\gamma_1}{4} - \frac{1}{4} \right) \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}^T \Theta_2 \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix} \right\} \end{aligned} \quad (234)$$

Since  $Q_4 \leq 0$  and  $\gamma_2 \in [-1, 1]$ , it follows that

$$-\gamma_2^4 x^T(t - \tau) Q_4 x(t - \tau) \leq -\gamma_2^2 x^T(t - \tau) Q_4 x(t - \tau) \quad (235)$$

Moreover, if  $x^T(t - \tau) Q_3 x(t - \tau) > 0$  then

$$-\gamma_2^3 x^T(t - \tau) Q_3 x(t - \tau) < \left( -\frac{3}{4}\gamma_2 + \frac{1}{4} \right) x^T(t - \tau) Q_3 x(t - \tau) \quad (236)$$

whereas if  $x^T(t - \tau) Q_3 x(t - \tau) < 0$  then

$$-\gamma_2^3 x^T(t - \tau) Q_3 x(t - \tau) < -\gamma_2^2 x^T(t - \tau) Q_3 x(t - \tau) \quad (237)$$

Therefore in either case,

$$-\gamma_2^3 x^T(t - \tau) Q_3 x(t - \tau) \leq \max \left\{ \left( -\frac{3}{4}\gamma_2 + \frac{1}{4} \right) x^T(t - \tau) Q_3 x(t - \tau), -\gamma_2^2 x^T(t - \tau) Q_3 x(t - \tau) \right\} \quad (238)$$

or that

$$-\gamma_2^3 x^T(t-\tau)Q_3x(t-\tau) \leq -(1-\beta)\gamma_2^2 x^T(t-\tau)Q_3x(t-\tau) - \beta(\frac{3}{4}\gamma_2 - \frac{1}{4})x^T(t-\tau)Q_3x(t-\tau)$$

where  $\beta \in \{0, 1\}$ . Collecting all previous results and substituting in (229), one obtains that

$$\dot{V}(x) \leq \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T M_4(\gamma_1, \gamma_2, \dot{\gamma}) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \quad (239)$$

where

$$M_4(\gamma_1, \gamma_2, \dot{\gamma}) = \gamma_1^2 N_2 + \gamma_1 N_1 + \gamma_2^2 N_3 + \gamma_2 N_4 + N_0(\dot{\gamma}) \quad (240)$$

and where  $N_0, N_1, N_2, N_3, N_4$  as in Eqs. (225)-(227). The inequalities  $N_3 \geq 0$  and  $N_2 \geq 0$  along with (224) and using Lemma 7.2 imply that  $M_4(\gamma_1, \gamma_2, \dot{\gamma}) < 0$  for all  $(\gamma_1, \gamma_2) \in [-1, 1] \times [-1, 1]$  and  $\dot{\gamma} \in [\underline{\dot{\gamma}}, \bar{\dot{\gamma}}]$ . The asymptotic stability of (210) then follows immediately from (239).  $\blacksquare$

**Remark 7.2** If  $A_2 = A_{d2} = 0$  and  $\dot{\gamma} \in (-\infty, +\infty)$ , it can be shown that condition (224) of Theorem 7.5 reduces to condition (212) of Theorem 7.2 with  $Q(\gamma) = Q_0 + \gamma Q_1$ .

In case the system matrices  $A(\gamma)$  and  $A_d(\gamma)$  depend only affinely on  $\gamma$  we have the following result.

**Theorem 7.6** *Consider the LPV time-delayed system (210) with*

$$A(\gamma) = A_0 + \gamma A_1, \quad A_d(\gamma) = A_{d0} + \gamma A_{d1} \quad (241)$$

where  $(\gamma, \dot{\gamma}) \in \mathcal{G} = [\underline{\gamma}, \bar{\gamma}] \times [\underline{\dot{\gamma}}, \bar{\dot{\gamma}}]$ . Assume that there exist a negative semi-definite matrix  $Q_2 \leq 0$ , a positive semi-definite matrix  $P_1 \geq 0$ , and symmetric matrices  $P_0, Q_0, Q_1$  which satisfy the following LMI's

$$Q(\gamma^\#) = Q_0 + \gamma^\# Q_1 + \gamma^{\#2} Q_2 > 0 \quad (242a)$$

$$P(\gamma^\#) = P_0 + \gamma^\# P_1 > 0 \quad (242b)$$



for all  $\gamma^\# \in \{\underline{\gamma}, \bar{\gamma}\}$  and

$$\gamma_1^{\#2} L_2 + \gamma_1^\# L_1 + \gamma_2^{\#2} L_3 + \gamma_2^\# L_4 + L_0(\nu) < 0 \quad (243)$$

for all  $\nu \in [\underline{\dot{\gamma}}, \bar{\dot{\gamma}}]$  and  $\gamma_i \in [\underline{\gamma}, \bar{\gamma}]$ ,  $i = 1, 2$ , and where

$$L_2 = \begin{bmatrix} P_1 A_1 + A_1^T P_1 + Q_2 + P_1 & 0 \\ 0 & A_{d1}^T P_1 A_{d1} \end{bmatrix} \geq 0 \quad (244)$$

$$L_1 = \begin{bmatrix} P_1 A_0 + P_0 A_1 + (\cdot)^T + Q_1 & P_1 A_{d0} + P_0 A_{d1} \\ A_{d0}^T P_1 + A_{d1}^T P_0 & 0 \end{bmatrix} \quad (245)$$

$$L_3 = \begin{bmatrix} 0 & 0 \\ 0 & -Q_2 \end{bmatrix} \geq 0 \quad (246)$$

$$L_4 = \begin{bmatrix} 0 & 0 \\ 0 & -Q_1 \end{bmatrix} \quad (247)$$

$$L_0(\dot{\gamma}) = \begin{bmatrix} P_0 A_0 + A_0^T P_0 + Q_0 + \dot{\gamma} P_1 & P_0 A_{d0} \\ A_{d0}^T P_0 & -Q_0 \end{bmatrix} \quad (248)$$

Then the system (210) is delay-independent stable for all  $(\gamma, \dot{\gamma}) \in \mathcal{G}$ .

*Proof.* Consider the following Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t) P(\gamma(t)) x(t) + \int_{-\tau}^0 x^T(t + \theta) Q(\gamma(t + \theta)) x(t + \theta) d\theta$$

where  $P(\gamma) = P_0 + \gamma P_1$  and  $Q(\gamma) = Q_0 + \gamma Q_1 + \gamma^2 Q_2$ . Since  $Q_2 \leq 0$ , and from (242a) and Lemma 7.1 it follows that  $Q(\gamma) > 0$  for all  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ . From (242b), it also follows that  $P(\gamma) > 0$  for all  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ . Therefore,  $V(t, x_t)$  is a positive definite functional with an infinitesimal upper bound. Calculation of the derivative of  $V$  yields

$$\begin{aligned} \dot{V}(t) = & \gamma_1 \{ 2x^T(t) [P_1 A_0 + P_0 A_1 + 0.5 Q_1] x(t) + 2x^T(t) [P_1 A_{d0} + P_0 A_{d1}] x(t - \tau) \} \\ & + \gamma_1^2 \{ 2x^T(t) [P_1 A_1 + 0.5 Q_2] x(t) + 2x^T(t) P_1 A_{d1} x(t - \tau) \} \end{aligned} \quad (249)$$

$$\begin{aligned} & - \gamma_2 x^T(t - \tau) Q_1 x(t - \tau) - \gamma_2^2 x^T(t - \tau) Q_2 x(t - \tau) \\ & + 2x^T(t) [P_0 A_0 + 0.5 Q_0 + 0.5 \dot{\gamma} P_1] x(t) \\ & + 2x^T(t) P_0 A_{d0} x^T(t - \tau) - x^T(t - \tau) Q_0 x(t - \tau) \end{aligned} \quad (250)$$

where  $\gamma_1 = \gamma(t)$  and  $\gamma_2 = \gamma(t - \tau)$ . Since  $P_1 > 0$  it follows that

$$2x^T(t)P_1A_{d1}x(t - \tau) \leq x^T(t)P_1x(t) + x^T(t - \tau)A_{d1}^TP_1A_{d1}x(t - \tau) \quad (251)$$

Substituting (251) in (249) and collecting terms yields

$$\dot{V} \leq \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}^T M_5(\gamma_1, \gamma_2, \dot{\gamma}) \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix} \quad (252)$$

where

$$M_5(\gamma_1, \gamma_2, \dot{\gamma}) = \gamma_1^2 L_2 + \gamma_1 L_1 + \gamma_2^2 L_3 + \gamma_2 L_4 + L_0(\dot{\gamma}) \quad (253)$$

where  $L_0, L_1, L_2, L_3, L_4$  as in (244)-(248). Since  $L_2 \geq 0$  and  $L_3 \geq 0$  then (243) implies, using Lemma 7.1, that  $M_5(\gamma_1, \gamma_2, \nu) < 0$  for all  $\nu \in [\underline{\dot{\gamma}}, \bar{\dot{\gamma}}]$ ,  $\gamma_i \in [\underline{\gamma}, \bar{\gamma}]$ ,  $i = 1, 2$ . Thus, the derivative of  $V$  is negative definite and the system (210) is asymptotically stable for all  $(\gamma, \dot{\gamma}) \in \mathcal{G}$  and  $\tau \in [0, \infty)$ . ■

**Remark 7.3** It can be shown that the condition (243) of Theorem 7.6 reduces to condition (212) of Theorem 7.2 when  $P_1 = Q_2 = 0$ .

It should be noted that in the previous results no additional conservatism is introduced by treating  $\gamma(t)$  and  $\gamma(t - \tau)$  as independent, even if the bound on  $\dot{\gamma}$  is arbitrarily small. However, for  $\dot{\gamma} = 0$ , one has that  $\gamma(t) = \gamma(t - \tau)$  for all  $t \geq 0$  and in this special case  $\gamma(t)$  and  $\gamma(t - \tau)$  are related (they are, in fact, equal). One possible method to account for the dependence of  $\gamma(t)$  and  $\gamma(t - \tau)$  is to eliminate  $\gamma(t - \tau)$  using the fact that  $\gamma(t) = \gamma(t - \tau) + \dot{\gamma}(\xi)\tau$ , for some  $\xi \in [t - \tau, t]$  and then take into account any known bounds for  $\dot{\gamma}$ . The resulting stability tests are then *delay-dependent*. We do not investigate this approach further since it follows from the previous results in a straightforward manner. Generally speaking, for small variation rates, a delay-dependent stability test should be used.

Next, we present several delay-dependent stability results for LPV systems, albeit without consideration to parameter variation rates. The latter problem is left for future investigation.

## 7.2 Delay-Dependent Stability

Herein, we derive stability conditions for the system (210) that take explicitly into account the delay bound  $\bar{\tau}$ . Before we give the main results, we re-write system (210) in the following equivalent forms.

$$\begin{aligned}\dot{x}(t) = & [A(\gamma(t)) + A_d(\gamma(t))]x(t) \\ & - A_d(\gamma(t)) \int_{-\tau}^0 [A(\gamma(t+\alpha))x(\alpha+t) + A_d(\gamma(t+\alpha))x(\alpha+t-\tau)]d\alpha\end{aligned}\quad (254)$$

and

$$\begin{aligned}\dot{x}(t) = & [A(\gamma(t)) + M A_d(\gamma(t))]x(t) + (I - M)A_d(\gamma(t))x(t - \tau) \\ & - M A_d(\gamma(t)) \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha\end{aligned}\quad (255)$$

where  $M$  in system (255) is an arbitrary matrix. The previous system transformation is similar to the one presented in [57] and [93]. Reference [93] used (255) to obtain stability tests for LTI time-delayed systems using frequency domain techniques. These results provided the exact counterpart of similar stability conditions developed in the time-domain [77, 78, 48, 60]. Most importantly, this frequency domain framework later allowed the derivation of new stability criteria with a very low degree of conservatism [89, 92, 91]. To see how systems (210) and (254) are related, notice that

$$x(t) - x(t - \tau) = \int_{-\tau}^0 \dot{x}(t + \alpha) d\alpha \quad (256)$$

Then

$$\begin{aligned}\dot{x}(t) = & A(\gamma(t)) x(t) + A_d(\gamma(t)) x(t - \tau) + A_d(\gamma(t)) x(t) - A_d(\gamma(t)) x(t) \\ = & [A(\gamma(t)) + A_d(\gamma(t))] x(t) - A_d(\gamma(t)) [x(t) - x(t - \tau)] \\ = & [A(\gamma(t)) + A_d(\gamma(t))] x(t) - A_d(\gamma(t)) \int_{-\tau}^0 \dot{x}(t + \alpha) d\alpha \\ = & [A(\gamma(t)) + A_d(\gamma(t))] x(t) \\ - & A_d(\gamma(t)) \int_{-\tau}^0 [A(\gamma(\alpha+t))x(\alpha+t) + A_d(\gamma(\alpha+t))x(\alpha+t-\tau)]d\alpha\end{aligned}$$

Similarly, for system (255), one has

$$\begin{aligned}
\dot{x}(t) &= A(\gamma(t)) x(t) + A_d(\gamma(t)) x(t - \tau) + MA_d(\gamma(t))[x(t) - x(t - \tau) - \int_{t-\tau}^t \dot{x}(\alpha) d\alpha] \\
&= A(\gamma(t)) x(t) + A_d(\gamma(t)) x(t - \tau) + MA_d(\gamma(t))x(t) - MA_d(\gamma(t))x(t - \tau) \\
&\quad - MA_d(\gamma(t)) \int_{t-\tau}^t \dot{x}(\alpha) d\alpha \\
&= [A(\gamma(t)) + MA_d(\gamma(t))]x(t) + (I - M)A_d(\gamma(t))x(t - \tau) \\
&\quad - MA_d(\gamma(t)) \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha
\end{aligned}$$

It follows that the trajectories of (254) or (255) are also trajectories of (210). Hence if systems (254) or (255) are stable, the original system (210) is also stable. It should be pointed out, however, that system (210) and systems (254) or (255) are not equivalent. Systems (254) and (255) include additional dynamics arising due to the eigenvalues of the matrix  $A_d$  [33]. Hence, any stability test based on either of these two systems is conservative.

### 7.2.1 Delay-Dependent Stability Conditions in LMI Form

Our first delay-dependent result is given by the following theorem.

**Theorem 7.7** *Let  $\gamma \in \Gamma = [\underline{\gamma}, \bar{\gamma}]$ . Then the system (254) is asymptotically stable for any constant delay  $\tau$ , with  $0 \leq \tau \leq \bar{\tau}$ , if there exist positive-definite matrices  $P, Q_1, Q_2$  such that the following matrix inequality is satisfied*

$$\begin{bmatrix}
A(\gamma)^T P + PA(\gamma) - Q_2 & -Q_2 & Q_2 + PA_d(\gamma) & \bar{\tau} A^T(\gamma) Q_1 & 0 \\
-Q_2 & -(Q_1 + Q_2) & Q_2 & 0 & 0 \\
A_d^T(\gamma) P + Q_2 & Q_2 & -Q_2 & 0 & \bar{\tau} A_d^T(\gamma) Q_2 \\
\bar{\tau} Q_1 A(\gamma) & 0 & 0 & -Q_1 & 0 \\
0 & 0 & \bar{\tau} Q_2 A_d(\gamma) & 0 & -Q_2
\end{bmatrix} < 0 \quad (257)$$

for all  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ .

*Proof.* Consider the following Lyapunov-Krasovskii functional  $V : \mathbb{R}_+ \times \mathcal{C}_{2\tau} \rightarrow \mathbb{R}_+$

$$\begin{aligned}
V(t, x_t) &= x^T(t) P x(t) + \int_{-\tau}^0 \int_{\beta}^0 [A(\gamma(t + \alpha))x(t + \alpha)]^T P_1 [A(\gamma(t + \alpha))x(t + \alpha)] d\alpha d\beta \\
&+ \int_{-\tau}^0 \int_{\beta}^0 [A_d(\gamma(t + \alpha))x(t + \alpha - \tau)]^T P_2 [A_d(\gamma(t + \alpha))x(t + \alpha - \tau)] d\alpha d\beta \quad (258)
\end{aligned}$$

Where  $P, P_1$  and  $P_2$  are constant, positive-definite matrices. According to Corollary 2.4,  $V$  is positive definite with an infinitesimal upper bound. The derivative of  $V$  along the trajectories of the system in Eq. (254) is

$$\begin{aligned}
\dot{V}(t) = & x^T(t)[(A(\gamma(t)) + A_d(\gamma(t)))^T P + P(A(\gamma(t)) + A_d(\gamma(t)))]x(t) \\
& - 2x^T(t)PA_d(\gamma(t)) \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha \\
& + \int_{-\tau}^0 [A(\gamma(t))x(t)]^T P_1[A(\gamma(t))x(t)] d\beta \\
& - \int_{-\tau}^0 [A(\gamma(t + \beta))x(t + \beta)]^T P_1[A(\gamma(t + \beta))x(t + \beta)] d\beta \\
& + \int_{-\tau}^0 [A_d(\gamma(t))x(t - \tau)]^T P_2[A_d(\gamma(t))x(t - \tau)] d\beta \\
& - \int_{-\tau}^0 [A_d(\gamma(t + \beta))x(t + \beta - \tau)]^T P_2[A_d(\gamma(t + \beta))x(t + \beta - \tau)] d\beta \quad (259)
\end{aligned}$$

or

$$\begin{aligned}
\dot{V}(t) = & x^T(t)[(A(\gamma(t)) + A_d(\gamma(t)))^T P + P(A(\gamma(t)) + A_d(\gamma(t)))]x(t) \\
& - 2x^T(t)PA_d(\gamma(t)) \int_{t-\tau}^t A(\gamma(\alpha))x(\alpha) d\alpha \\
& - 2x^T(t)PA_d(\gamma(t)) \int_{t-\tau}^t A_d(\gamma(\alpha))x(\alpha - \tau) d\alpha \\
& + \tau [A(\gamma(t))x(t)]^T P_1[A(\gamma(t))x(t)] - \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha)]^T P_1[A(\gamma(\alpha))x(\alpha)] d\alpha \\
& + \tau [A_d(\gamma(t))x(t - \tau)]^T P_2[A_d(\gamma(t))x(t - \tau)] \\
& - \int_{t-\tau}^t [A_d(\gamma(\alpha))x(\alpha - \tau)]^T P_2[A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha. \quad (260)
\end{aligned}$$

By the Cauchy-Schwartz inequality one can show that for any positive definite matrix  $P$  and any  $v(\alpha) \in \mathbb{R}^n$ ,

$$\tau \int_{t-\tau}^t v(\alpha)^T P v(\alpha) d\alpha \geq \left[ \int_{t-\tau}^t v(\alpha) d\alpha \right]^T P \left[ \int_{t-\tau}^t v(\alpha) d\alpha \right] \quad (261)$$

Using this result, and because  $0 \leq \tau \leq \bar{\tau}$ , we have

$$\begin{aligned}
& - \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha)]^T P_1[A(\gamma(\alpha))x(\alpha)] d\alpha \\
& \leq -\left(\frac{1}{\bar{\tau}}\right) \left[ \int_{t-\tau}^t A(\gamma(\alpha))x(\alpha) d\alpha \right]^T P_1 \left[ \int_{t-\tau}^t A(\gamma(\alpha))x(\alpha) d\alpha \right]
\end{aligned}$$

and

$$\begin{aligned}
& - \int_{t-\tau}^t [A_d(\gamma(\alpha))x(\alpha - \tau)]^T P_2 [A_d(\gamma(\alpha))x(\alpha - \tau)] \, d\alpha \\
& \leq -\left(\frac{1}{\bar{\tau}}\right) \left[ \int_{t-\tau}^t A_d(\gamma(\alpha))x(\alpha - \tau) \, d\alpha \right]^T P_2 \left[ \int_{t-\tau}^t A_d(\gamma(\alpha))x(\alpha - \tau) \, d\alpha \right]
\end{aligned}$$

Define now the new variables

$$y(t) = - \int_{t-\tau}^t A(\gamma(\alpha))x(\alpha) \, d\alpha \quad \text{and} \quad z(t) = - \int_{t-\tau}^t A_d(\gamma(\alpha))x(\alpha - \tau) \, d\alpha$$

Using (210) one obtains

$$z(t) = -x(t) + x(t - \tau) - y(t) \quad (262)$$

and because  $0 \leq \tau \leq \bar{\tau}$ , one obtains the inequality

$$\begin{aligned}
\dot{V} \leq & x^T(t) [(A(\gamma(t)) + A_d(\gamma(t)))^T P + P(A(\gamma(t)) + A_d(\gamma(t)))] x(t) \\
& + 2x^T(t) P A_d(\gamma(t)) y(t) + 2x^T(t) P A_d(\gamma(t)) z(t) \\
& + \bar{\tau} x^T(t) A^T(\gamma(t)) P_1 A(\gamma(t)) x(t) + \bar{\tau} x^T(t - \tau) A_d^T(\gamma(t)) P_2 A_d(\gamma(t)) x(t - \tau) \\
& - \left(\frac{1}{\bar{\tau}}\right) \left[ \int_{t-\tau}^t A(\gamma(\alpha))x(\alpha) \, d\alpha \right]^T P_1 \left[ \int_{t-\tau}^t A(\gamma(\alpha))x(\alpha) \, d\alpha \right] \\
& - \left(\frac{1}{\bar{\tau}}\right) \left[ \int_{t-\tau}^t A_d(\gamma(\alpha))x(\alpha - \tau) \, d\alpha \right]^T P_2 \left[ \int_{t-\tau}^t A_d(\gamma(\alpha))x(\alpha - \tau) \, d\alpha \right]
\end{aligned}$$

where we made use of Eq. (261). The previous inequality can be rewritten in the form,

$$\begin{aligned}
\dot{V} \leq & x^T(t) [(A(\gamma(t)) + A_d(\gamma(t)))^T P + P(A(\gamma(t)) + A_d(\gamma(t)))] x(t) + 2x^T(t) P A_d(\gamma(t)) y(t) \\
& + \bar{\tau} x^T(t - \tau) A_d^T(\gamma(t)) P_2 A_d(\gamma(t)) x(t - \tau) - y^T(t) (P_1 / \bar{\tau}) y(t) - z^T(t) (P_2 / \bar{\tau}) z(t) \\
& + 2x^T(t) P A_d(\gamma(t)) z(t) + \bar{\tau} x^T(t) A^T(\gamma(t)) P_1 A(\gamma(t)) x(t) \quad (263)
\end{aligned}$$

Substituting (262) into (263) and with  $\bar{X}(t) = [x(t), y(t), x(t - \tau)]^T$ , one obtains

$$\dot{V} \leq \bar{X}^T(t) \begin{bmatrix} \begin{pmatrix} A(\gamma(t))^T P + P A(\gamma(t)) \\ + \bar{\tau} A^T(\gamma(t)) P_1 A(\gamma(t)) \\ - P_2 / \bar{\tau} \\ - P_2 / \bar{\tau} \end{pmatrix} & -P_2 / \bar{\tau} & P_2 / \bar{\tau} + P A_d(\gamma(t)) \\ -P_1 / \bar{\tau} - P_2 / \bar{\tau} & P_2 / \bar{\tau} & \\ P_2 / \bar{\tau} + A_d^T(\gamma(t)) P & P_2 / \bar{\tau} & \begin{pmatrix} \bar{\tau} A_d^T(\gamma(t)) P_2 A_d(\gamma(t)) \\ - P_2 / \bar{\tau} \end{pmatrix} \end{bmatrix} \bar{X}(t) \quad (264)$$

Hence, if the following inequality is satisfied, the time-delayed system (210) will be asymptotically stable for any  $0 \leq \tau \leq \bar{\tau}$  and  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$

$$\begin{bmatrix} \begin{pmatrix} A(\gamma)^T P + P A(\gamma) \\ + \bar{\tau} A^T(\gamma) P_1 A(\gamma) - P_2/\bar{\tau} \end{pmatrix} & -P_2/\bar{\tau} & P_2/\bar{\tau} + P A_d(\gamma) \\ -P_2/\bar{\tau} & -P_1/\bar{\tau} - P_2/\bar{\tau} & P_2/\bar{\tau} \\ P_2/\bar{\tau} + A_d^T(\gamma) P & P_2/\bar{\tau} & \bar{\tau} A_d^T(\gamma) P_2 A_d(\gamma) - P_2/\bar{\tau} \end{bmatrix} < 0 \quad (265)$$

Inequality (265) can be rewritten as follows

$$\begin{bmatrix} A(\gamma)^T P + P A(\gamma) - P_2/\bar{\tau} & -P_2/\bar{\tau} & P_2/\bar{\tau} + P A_d(\gamma) & A^T(\gamma) P_1 & 0 \\ -P_2/\bar{\tau} & -P_1/\bar{\tau} - P_2/\bar{\tau} & P_2/\bar{\tau} & 0 & 0 \\ A_d^T(\gamma) P + P_2/\bar{\tau} & P_2/\bar{\tau} & -P_2/\bar{\tau} & 0 & A_d^T(\gamma) P_2 \\ P_1 A(\gamma) & 0 & 0 & -P_1/\bar{\tau} & 0 \\ 0 & 0 & P_2 A_d(\gamma) & 0 & -P_2/\bar{\tau} \end{bmatrix} < 0 \quad (266)$$

Let  $Q_1 = P_1/\bar{\tau}$ ,  $Q_2 = P_2/\bar{\tau}$ . Then, inequality (266) can be rewritten as (257), which has to be satisfied for all  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ . Since the parameter  $\gamma$  lies in a compact interval,  $\dot{V}$  is uniformly negative definite with respect to  $\gamma$  and system (210) is asymptotically stable. ■

Gridding of the interval  $[\underline{\gamma}, \bar{\gamma}]$  is required to reduce the infinite system of LMI's in (257) to a finite set of LMI's. Alternatively, for an LPV time-delayed system, for which the system matrices are affine functions of  $\gamma$ , the LMI (257) need only to be checked at the boundary points of the interval  $\Gamma = [\underline{\gamma}, \bar{\gamma}]$ . This statement is formalized in the following corollary.

**Corollary 7.1** *Consider the system (254) with*

$$A(\gamma) = A_0 + \gamma A_1, \quad A_d(\gamma) = A_{d0} + \gamma A_{d1}, \quad \gamma \in [\underline{\gamma}, \bar{\gamma}] \quad (267)$$

*Suppose that there exist positive-definite matrices  $P, Q_1, Q_2$  such that the following matrix inequality is satisfied for all  $\gamma^\# \in \{\underline{\gamma}, \bar{\gamma}\}$ , where  $H_{11} = A(\gamma^\#)^T P + P A(\gamma^\#) - Q_2$ .*

$$\begin{bmatrix} H_{11} & -Q_2 & Q_2 + P A_d(\gamma^\#) & \bar{\tau} A^T(\gamma^\#) Q_1 & 0 \\ -Q_2 & -Q_1 - Q_2 & Q_2 & 0 & 0 \\ A_d^T(\gamma^\#) P + Q_2 & Q_2 & -Q_2 & 0 & \bar{\tau} A_d^T(\gamma^\#) Q_2 \\ \bar{\tau} Q_1 A(\gamma^\#) & 0 & 0 & -Q_1 & 0 \\ 0 & 0 & \bar{\tau} Q_2 A_d(\gamma^\#) & 0 & -Q_2 \end{bmatrix} < 0 \quad (268)$$

Then (254) is asymptotically stable for any constant  $\tau$ , such that  $0 \leq \tau \leq \bar{\tau}$ , and all  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ .

Another delay-dependent stability test is given below. The approach is based on a generalization of the stability test in [61] to LPV systems.

**Theorem 7.8** *Consider the system (255) where  $0 \leq \tau \leq \bar{\tau}$ , and  $\gamma \in \Gamma = [\underline{\gamma}, \bar{\gamma}]$ . Suppose that there exist positive-definite matrices  $P, Z$  and  $Q$  such that for any constant matrix  $\bar{M}, \bar{W}$ , the following LMI condition is satisfied*

$$\begin{bmatrix} \begin{pmatrix} A(\gamma)^T P + A_d(\gamma)^T \bar{M}^T \\ + P A(\gamma) + \bar{M} A_d(\gamma) \\ + Q + A(\gamma)^T Z A_d(\gamma) \end{pmatrix} & \bar{\tau} \bar{M} A_d(\gamma) + \bar{W} & \begin{pmatrix} P A_d(\gamma) - \bar{M} A_d(\gamma) \\ + A(\gamma)^T Z A_d(\gamma) \end{pmatrix} & 0 \\ * & \begin{pmatrix} -Z + \bar{\tau} \bar{W} \\ + \bar{\tau} \bar{W}^T \end{pmatrix} & -\bar{W}^T & \bar{\tau} \bar{W}^T \\ * & * & -Q + A_d(\gamma)^T Z A_d(\gamma) & 0 \\ * & * & * & -Z \end{bmatrix} < 0 \quad (269)$$

for all  $\gamma \in \Gamma$ . Then (255) is asymptotically stable for all  $\tau \in [0, \bar{\tau}]$ .

Before we proceed with the proof of this theorem, we present a lemma that will be used later on.

**Lemma 7.3 ([60])** *For any  $X > 0$  and any  $W$ , and  $\Sigma = (W^T X + I)X^{-1}(XW + I)$  the following inequality holds*

$$-2 \int_{\Omega} b^T(\alpha) a(\alpha) d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & XW \\ W^T X & \Sigma \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} d\alpha \quad (270)$$

*Proof.* [of Theorem 7.8] Consider the Lyapunov-Krasovskii functional  $V : \mathbb{R}_+ \times \mathcal{C}_{2\tau} \rightarrow \mathbb{R}_+$  given by

$$\begin{aligned} V(t, x_t) &= x^T(t) P x(t) + \int_{-\tau}^0 x^T(t + \alpha) Q x(t + \alpha) d\alpha + \\ &\int_{-\tau}^0 \int_{\beta}^0 [A(\gamma(t + \alpha)) x(t + \alpha) + A_d(\gamma(t + \alpha)) x(t + \alpha - \tau)]^T Y \\ &\times [A(\gamma(t + \alpha)) x(t + \alpha) + A_d(\gamma(t + \alpha)) x(t + \alpha - \tau)] d\alpha d\beta \end{aligned}$$



Where  $P, Q, Y$  are positive-definite matrices. According to Corollary 2.4,  $V$  is positive definite and has an infinitesimal upper bound. Taking the derivative of  $V$  along the trajectories of the system (255) which is equivalent to system (210), one obtains,

$$\begin{aligned}
\dot{V}(t) &= x^T(t)[(A(\gamma(t)) + MA_d(\gamma(t)))^T P + P(A(\gamma(t)) + MA_d(\gamma(t)))]x(t) \\
&+ x^T(t)[P(I - M)A_d(\gamma(t))]x(t - \tau) + x^T(t - \tau)[P(I - M)A_d(\gamma(t))]^T x(t) \\
&- 2x^T(t)PMA_d(\gamma(t)) \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha \\
&+ x^T(t)Qx(t) - x^T(t - \tau)Qx(t - \tau) \\
&+ \int_{-\tau}^0 [A(\gamma(t))x(t) + A_d(\gamma(t))x(t - \tau)]^T Y [A(\gamma(t))x(t) + A_d(\gamma(t))x(t - \tau)] d\beta \\
&- \int_{-\tau}^0 [A(\gamma(t + \beta))x(t + \beta) + A_d(\gamma(t + \beta))x(t + \beta - \tau)]^T Y \\
&\quad \times [A(\gamma(t + \beta))x(t + \beta) + A_d(\gamma(t + \beta))x(t + \beta - \tau)] d\beta
\end{aligned}$$

The last integral can be written as

$$\begin{aligned}
&\int_{-\tau}^0 [A(\gamma(t + \beta))x(t + \beta) + A_d(\gamma(t + \beta))x(t + \beta - \tau)]^T Y \\
&\quad \times [A(\gamma(t + \beta))x(t + \beta) + A_d(\gamma(t + \beta))x(t + \beta - \tau)] d\beta \\
&= \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)]^T Y \\
&\quad \times [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha
\end{aligned} \tag{271}$$

Hence,

$$\begin{aligned}
\dot{V}(t) &\leq x^T(t)[(A(\gamma(t)) + MA_d(\gamma(t)))^T P + P(A(\gamma(t)) + MA_d(\gamma(t)))]x(t) \\
&+ x^T(t)[P(I - M)A_d(\gamma(t))]x(t - \tau) + x^T(t - \tau)[P(I - M)A_d(\gamma(t))]^T x(t) \\
&- 2x^T(t)PMA_d(\gamma(t)) \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha \\
&+ x^T(t)Qx(t) - x^T(t - \tau)Qx(t - \tau) \\
&+ \bar{\tau}[A(\gamma(t))x(t) + A_d(\gamma(t))x(t - \tau)]^T Y [A(\gamma(t))x(t) + A_d(\gamma(t))x(t - \tau)] \\
&- \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)]^T Y \\
&\quad \times [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha
\end{aligned} \tag{272}$$

Let now  $a(\alpha) = [A(\gamma)x(\alpha) + A_d(\gamma)x(\alpha - \tau)]$  and  $b(\alpha) = A_d^T(\gamma(t))M^T Px(t)$  and use

Lemma 7.3 to obtain

$$\begin{aligned}
& -2x^T(t)PMA_d(\gamma(t)) \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha \\
& \leq \tau x^T(t)PMA_d(\gamma(t))[W^TY + I]Y^{-1}[YW + I]A_d^T(\gamma(t))M^TPx(t) \\
& + 2x^T(t)PMA_d(\gamma(t))W^TY \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha \\
& + \int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)]^TY[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha
\end{aligned} \tag{273}$$

for any matrix  $W$ . Using the fact

$$\int_{t-\tau}^t [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha - \tau)] d\alpha = \int_{t-\tau}^t \dot{x}(\alpha) d\alpha = x(t) - x(t - \tau) \tag{274}$$

and substituting (273) and (274) into (272), we have

$$\begin{aligned}
\dot{V}(t) & \leq x^T(t)[(A(\gamma(t)) + MA_d(\gamma(t)))^TP + P(A(\gamma(t)) + MA_d(\gamma(t)))]x(t) \\
& + x^T(t)[P(I - M)A_d(\gamma(t))]x(t - \tau) + x^T(t - \tau)[P(I - M)A_d(\gamma(t))]^Tx(t) \\
& + \bar{\tau}x^T(t)PMA_d(\gamma(t))[W^TY + I]Y^{-1}[YW + I]A_d^T(\gamma(t))M^TPx(t) \\
& + 2x^T(t)PMA_d(\gamma(t))W^TY[x(t) - x(t - \tau)] \\
& + x^T(t)Qx(t) - x^T(t - \tau)Qx(t - \tau) \\
& + \bar{\tau}[A(\gamma(t))x(t) + A_d(\gamma(t))x(t - \tau)]^TY[A(\gamma(t))x(t) + A_d(\gamma(t))x(t - \tau)]
\end{aligned} \tag{275}$$

In the above inequality,  $W$  is an arbitrary matrix. Noticing that

$$\begin{aligned}
& \bar{\tau}x^T(t)PMA_d(\gamma)[W^TY + I]Y^{-1}[YW + I]A_d^T(\gamma)M^TPx(t) \\
= & \bar{\tau}x^T(t)PMA_d(\gamma)[W^TYW + W^T + W - Y^{-1}]A_d^T(\gamma)M^TPx(t) \\
& + 2\bar{\tau}x^T(t)PMA_d(\gamma)Y^{-1}A_d^T(\gamma)M^TPx(t) \\
= & \bar{\tau}x^T(t)PMA_d(\gamma)Y^{-1}[YW^TY Y^{-1}YWY + YW^TY + YWY - Y]Y^{-1}A_d^T(\gamma)M^TPx(t) \\
& + 2\bar{\tau}x^T(t)PMA_d(\gamma)Y^{-1}A_d^T(\gamma)M^TPx(t) \\
= & \bar{\tau}y^T(t)[\bar{W}^TY^{-1}\bar{W} + \bar{W}^T + \bar{W} - Y]y(t) + 2\bar{\tau}x^T(t)PMA_d(\gamma)y(t)
\end{aligned} \tag{276}$$

where  $y(t)$  and  $\bar{W}$  are defined by

$$y(t) = Y^{-1}A_d^T(\gamma)M^TPx(t), \quad \bar{W} = YWY$$

because  $W$  is an arbitrary matrix,  $\bar{W}$  is also an arbitrary matrix, and

$$2x^T(t)PMA_d(\gamma)W^TY[x(t) - x(t - \tau)] = 2y^T(t)\bar{W}^T[x(t) - x(t - \tau)] \quad (277)$$

and substituting (276) and (277) into (275), one obtains

$$\begin{aligned} \dot{V}(t) &\leq x^T(t)[(A(\gamma) + MA_d(\gamma))^TP + P(A(\gamma) + MA_d(\gamma)) + Q]x(t) \\ &\quad + 2x^T(t)[P(I - M)A_d(\gamma)]x(t - \tau) - x^T(t - \tau)Qx(t - \tau) \\ &\quad + \bar{\tau}y^T(t)[\bar{W}^TY^{-1}\bar{W} + \bar{W}^T + \bar{W} - Y]y(t) + 2\bar{\tau}x^T(t)PMA_d(\gamma)y(t) \\ &\quad + 2y^T(t)\bar{W}^T[x(t) - x(t - \tau)] \\ &\quad + \bar{\tau}[A(\gamma)x(t) + A_d(\gamma)x(t - \tau)]^TY[A(\gamma)x(t) + A_d(\gamma)x(t - \tau)] \end{aligned} \quad (278)$$

Let now  $\bar{X}(t) = [x(t) \ y(t) \ x(t - \tau)]^T$ . Inequality (278) can be rewritten as follows:

$$\dot{V}(t) \leq \bar{X}^T(t) \begin{bmatrix} \begin{pmatrix} (A(\gamma) + MA_d(\gamma))^TP \\ +P(A(\gamma) + MA_d(\gamma)) \\ +Q + A(\gamma)^T\bar{\tau}YA(\gamma) \end{pmatrix} & \bar{\tau}PMA_d(\gamma) + \bar{W} & \begin{pmatrix} P(I - M)A_d(\gamma) \\ +A(\gamma)^T\bar{\tau}YA_d(\gamma) \end{pmatrix} \\ * & \begin{pmatrix} -Y\bar{\tau} + \bar{\tau}\bar{W} \\ +\bar{\tau}\bar{W}^T \\ +\bar{\tau}\bar{W}^TY^{-1}\bar{W} \end{pmatrix} & -\bar{W}^T \\ * & * & -Q + A_d(\gamma)^T\bar{\tau}YA_d(\gamma) \end{bmatrix} \bar{X}(t) \quad (279)$$

Using Schur complements, the matrix in the right hand side of (279) is negative definite if

$$\begin{bmatrix} \begin{pmatrix} (A(\gamma) + MA_d(\gamma))^TP \\ +P(A(\gamma) + MA_d(\gamma)) \\ +Q + A(\gamma)^T\bar{\tau}YA(\gamma) \end{pmatrix} & \bar{\tau}PMA_d(\gamma) + \bar{W} & \begin{pmatrix} P(I - M)A_d(\gamma) \\ +A(\gamma)^T\bar{\tau}YA_d(\gamma) \end{pmatrix} & 0 \\ * & \begin{pmatrix} -Y\bar{\tau} + \bar{\tau}\bar{W} \\ +\bar{\tau}\bar{W}^T \end{pmatrix} & -\bar{W}^T & \bar{\tau}\bar{W}^T \\ * & * & -Q + A_d(\gamma)^T\bar{\tau}YA_d(\gamma) & 0 \\ * & * & * & -\bar{\tau}Y \end{bmatrix} < 0 \quad (280)$$

Let  $Z = \bar{\tau}Y$  and  $\bar{M} = PM$ . Then (280) is equivalent to (269). Because  $M$  is an arbitrary matrix and  $P$  is positive definite,  $\bar{M}$  is also an arbitrary matrix. If LMI (269) is satisfied for all  $\gamma \in \Gamma$ , the derivative of  $V$  is uniformly negative definite and the system (210) is asymptotically stable. This completes the proof of the theorem.  $\blacksquare$

**Remark 7.4** It can be shown that – when restricted to LTI systems – (269) is slightly more general than the condition in [61] assuming that  $A_d$  is invertible. This is due to the extra variable  $\bar{W}$ . In fact, it can be shown [93] that (269) reduces to the condition in [61] when  $\bar{W} = 0$ . In case  $A_d$  is not invertible, condition (269) (with  $\bar{W} = 0$ ) is implied by the condition in [93].

Condition (269) is an infinite-dimensional set of LMI's in the unknowns  $P, \bar{M}, \bar{W}, Z$  and  $Q$ . These LMI's can be checked by gridding the parameter space. As before, gridding can be avoided in case the system matrices are affine functions of the parameter  $\gamma$ .

**Corollary 7.2** *Consider the system (255) with*

$$A(\gamma) = A_0 + \gamma A_1, \quad A_d(\gamma) = A_{d0} + \gamma A_{d1}, \quad \gamma \in [\underline{\gamma}, \bar{\gamma}] \quad (281)$$

*Suppose there exist positive-definite matrices  $P, Q$  and  $Z$  such that for any constant matrix  $\bar{M}, \bar{W}$ , the following matrix inequality is satisfied*

$$\begin{bmatrix} \begin{pmatrix} A(\gamma^\#)^T P + A_d(\gamma^\#)^T \bar{M}^T \\ + P A(\gamma^\#) + \bar{M} A_d(\gamma^\#) \\ + Q + A(\gamma^\#)^T Z A(\gamma^\#) \end{pmatrix} & \bar{\tau} \bar{M} A_d(\gamma^\#) + \bar{W} & \begin{pmatrix} P A_d(\gamma^\#) - \bar{M} A_d(\gamma^\#) \\ + A(\gamma^\#)^T Z A_d(\gamma^\#) \end{pmatrix} & 0 \\ * & \begin{pmatrix} -Z + \bar{\tau} \bar{W} \\ + \bar{\tau} \bar{W}^T \end{pmatrix} & -\bar{W}^T & \bar{\tau} \bar{W}^T \\ * & * & -Q + A_d(\gamma^\#)^T Z A_d(\gamma^\#) & 0 \\ * & * & * & -Z \end{bmatrix} < 0 \quad (282)$$

*where  $\gamma^\# \in \{\underline{\gamma}, \bar{\gamma}\}$ . Then (210) is asymptotically stable for any constant delay  $\tau$ , such that  $0 \leq \tau \leq \bar{\tau}$ .*

The third delay-dependent stability result for (210) is given next.

**Theorem 7.9** *Consider the system (210) with  $\gamma \in \Gamma = [\underline{\gamma}, \bar{\gamma}]$ . If there exist positive-definite matrices  $Q_1, Q_2, Q, Z$  and  $P$ , for any constant matrix  $R, \bar{W}$ , such that the following*

condition is satisfied for  $\gamma \in \Gamma$

$$\begin{bmatrix} H_{11}(\gamma) & -Q_2 & \bar{\tau}RA_d(\gamma) + \bar{W} & H_{14}(\gamma) & \bar{\tau}A^T(\gamma)Q_1 & 0 & 0 \\ * & -(Q_1 + Q_2) & 0 & Q_2 & 0 & 0 & 0 \\ * & * & H_{33}(\gamma) & -\bar{W}^T & 0 & 0 & \bar{\tau}\bar{W} \\ * & * & * & H_{44}(\gamma) & 0 & \bar{\tau}A^T(\gamma)Q_2 & 0 \\ * & * & * & * & -Q_1 & 0 & 0 \\ * & * & * & * & * & -Q_2 & 0 \\ & * & * & * & * & * & -Z \end{bmatrix} < 0 \quad (283)$$

where,

$$\begin{aligned} H_{11}(\gamma) &= 2A^T(\gamma)P + A_d^T(\gamma)R^T + 2PA(\gamma) + RA_d(\gamma) + Q - Q_2 + A^T(\gamma)ZA(\gamma) \\ H_{14}(\gamma) &= PA_d(\gamma) + Q_2 - RA_d(\gamma) + A^T(\gamma)ZA_d(\gamma) \\ H_{33}(\gamma) &= -Z + \bar{\tau}\bar{W} + \bar{\tau}\bar{W}^T \\ H_{44}(\gamma) &= -Q - Q_2 + A_d^T(\gamma)ZA_d(\gamma) \end{aligned} \quad (284)$$

then (210) is asymptotically stable for any constant delay  $\tau \in [0, \bar{\tau}]$ .

*Proof.* Consider the following positive-definite functional,

$$\begin{aligned} V(t, x_t) &= 2x^T(t)Px(t) + \int_{-\tau}^0 \int_{\beta}^0 [A(\gamma(t+\alpha))x(t+\alpha)]^T P_1 [A(\gamma(t+\alpha))x(t+\alpha)] d\alpha d\beta \\ &+ \int_{-\tau}^0 \int_{\beta}^0 [A_d(\gamma(t+\alpha))x(t+\alpha-\tau)]^T P_2 [A_d(\gamma(t+\alpha))x(t+\alpha-\tau)] d\alpha d\beta \\ &+ \int_{-\tau}^0 x^T(t+\alpha)Qx(t+\alpha) d\alpha \\ &+ \int_{-\tau}^0 \int_{\beta}^0 [A(\gamma(t+\alpha))x(t+\alpha) + A_d(\gamma(t+\alpha))x(t+\alpha-\tau)]^T Y \\ &\quad \times [A(\gamma(t+\alpha))x(t+\alpha) + A_d(\gamma(t+\alpha))x(t+\alpha-\tau)] d\alpha d\beta \end{aligned}$$

where  $P, P_1, P_2, Q, Y$  are constant positive-definite matrices. This  $V(t, x_t)$  is a positive definite functional and has an infinitesimal upper bound according to the Lemma 2.5. Taking now the derivative of  $V$  along the trajectories of the system (254) or (255) and defining

$$\begin{aligned} y(t) &= -\int_{t-\tau}^t A(\gamma(\alpha))x(\alpha) d\alpha, & g(t) &= Y^{-1}A_d^T(\gamma(t))M^T Px(t) \\ \bar{X}_1 &= \begin{bmatrix} x(t) & y(t) & x(t-\tau) \end{bmatrix}^T, & \bar{X}_2 &= \begin{bmatrix} x(t) & g(t) & x(t-\tau) \end{bmatrix}^T \end{aligned}$$

one obtains,

$$\dot{V}(t) \leq \bar{X}_1^T \begin{bmatrix} \begin{pmatrix} A(\gamma)^T P + P A(\gamma) \\ + \bar{\tau} A^T(\gamma) P_1 A(\gamma) \\ - P_2 / \bar{\tau} \end{pmatrix} & -P_2 / \bar{\tau} & P_2 / \bar{\tau} + P A_d(\gamma) \\ -P_2 / \bar{\tau} & \begin{pmatrix} -P_1 / \bar{\tau} \\ -P_2 / \bar{\tau} \end{pmatrix} & P_2 / \bar{\tau} \\ P_2 / \bar{\tau} + A_d^T(\gamma) P & P_2 / \bar{\tau} & \begin{pmatrix} \bar{\tau} A_d^T(\gamma) P_2 A_d(\gamma) \\ -P_2 / \bar{\tau} \end{pmatrix} \end{bmatrix} \bar{X}_1 +$$

$$\bar{X}_2^T \begin{bmatrix} \begin{pmatrix} (A(\gamma) + M A_d(\gamma))^T P \\ + P(A(\gamma) + M A_d(\gamma)) \\ + Q + A(\gamma)^T \bar{\tau} Y A(\gamma) \end{pmatrix} & \bar{\tau} P M A_d(\gamma) + \bar{W} & \begin{pmatrix} P(I - M) A_d(\gamma) \\ + A(\gamma)^T \bar{\tau} Y A_d(\gamma) \end{pmatrix} \\ * & \begin{pmatrix} -Y \bar{\tau} + \bar{\tau} \bar{W} \\ + \bar{\tau} \bar{W}^T \\ + \bar{\tau} \bar{W}^T Y^{-1} \bar{W} \end{pmatrix} & -\bar{W}^T \\ * & * & \begin{pmatrix} -Q + \\ A_d(\gamma)^T \bar{\tau} Y A_d(\gamma) \end{pmatrix} \end{bmatrix} \bar{X}_2$$

which can be rewritten as  $\dot{V}(x(t)) \leq \bar{X}_3^T [\Sigma] \bar{X}_3$ , where  $\bar{X}_3 = \begin{bmatrix} x(t) & y(t) & g(t) & x(t - \tau) \end{bmatrix}^T$

and

$$[\Sigma] = \begin{bmatrix} \begin{pmatrix} (A(\gamma) + M A_d(\gamma))^T P \\ + P(A(\gamma) + M A_d(\gamma)) \\ + Q + \bar{\tau} A^T(\gamma) Y A(\gamma) \\ A^T(\gamma) P + P A(\gamma) \\ + \bar{\tau} A^T(\gamma) P_1 A(\gamma) \\ - P_2 / \bar{\tau} \end{pmatrix} & -P_2 / \bar{\tau} & \bar{\tau} P M A_d(\gamma) + \bar{W} & \begin{pmatrix} P A_d(\gamma) + P_2 / \bar{\tau} \\ + P(I - M) A_d(\gamma) \\ \bar{\tau} A^T(\gamma) Y A_d(\gamma) \end{pmatrix} \\ * & -(P_1 + P_2) / \bar{\tau} & 0 & P_2 / \bar{\tau} \\ * & * & \begin{pmatrix} -Y \bar{\tau} + \bar{\tau} \bar{W} \\ + \bar{\tau} \bar{W}^T \\ + \bar{\tau} \bar{W}^T Y^{-1} \bar{W} \end{pmatrix} & -\bar{W}^T \\ * & * & * & \begin{pmatrix} -Q - P_2 / \bar{\tau} \\ + \bar{\tau} A_d^T(\gamma) P_2 A_d(\gamma) \\ + \bar{\tau} A_d^T(\gamma) Y A_d(\gamma) \end{pmatrix} \end{bmatrix}$$

The stability of system (210) can be guaranteed by satisfying  $[\Sigma] < 0$ . This inequality is

not an LMI. However, using  $Q_1 = P_1/\bar{\tau}$ ,  $Q_2 = P_2/\bar{\tau}$ ,  $R = PM$ ,  $Z = \bar{\tau}Y$  and using the Schur complement theorem, the requirement for  $[\Sigma]$  to be negative definite is equivalent to (283) which is an LMI.  $\blacksquare$

The solution of the infinite-dimensional set of matrix inequalities in (283) can be checked by gridding the parameter space. As with Theorems 7.7 and 7.8 gridding can be avoided if the state matrices are an affine function of the parameter.

**Corollary 7.3** *Consider system (255) where*

$$A(\gamma) = A_0 + \gamma A_1, \quad A_d(\gamma) = A_{d0} + \gamma A_{d1}, \quad \gamma \in [\underline{\gamma}, \bar{\gamma}] \quad (285)$$

*Suppose there exist constant positive-definite matrices  $Q_1, Q_2, Q, Z, P$ , and any constant matrix  $R, \bar{W}$ , such that the following condition is satisfied*

$$\begin{bmatrix} H_{11}(\gamma^\#) & -Q_2 & \bar{\tau}RA_d(\gamma^\#) + \bar{W} & H_{14}(\gamma^\#) & \bar{\tau}A^T(\gamma^\#)Q_1 & 0 & 0 \\ * & -(Q_1 + Q_2) & 0 & Q_2 & 0 & 0 & 0 \\ * & * & H_{33}(\gamma^\#) & -\bar{W}^T & 0 & 0 & \bar{\tau}\bar{W} \\ * & * & * & H_{44}(\gamma^\#) & 0 & \bar{\tau}A^T(\gamma^\#)Q_2 & 0 \\ * & * & * & * & -Q_1 & 0 & 0 \\ * & * & * & * & * & -Q_2 & 0 \\ & * & * & * & * & * & -Z \end{bmatrix} < 0 \quad (286)$$

*for all  $\gamma^\# \in \{\underline{\gamma}, \bar{\gamma}\}$ , with  $H_{11}(\gamma)$ ,  $H_{14}(\gamma)$ ,  $H_{33}(\gamma)$  and  $H_{44}(\gamma)$  as in (284). Then (255) is asymptotically stable for all constant delays  $\tau \in [0, \bar{\tau}]$ .*

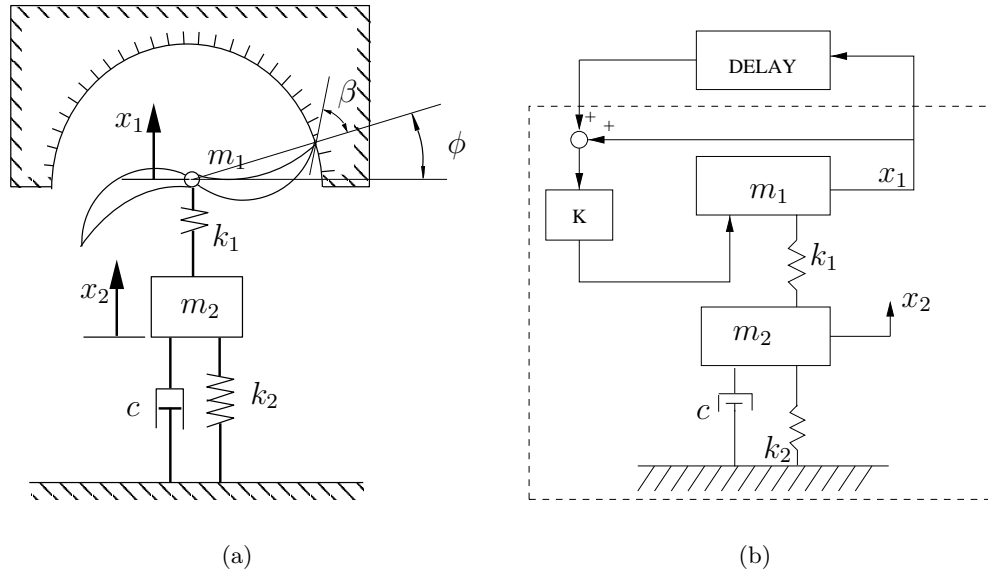
*Proof.* It suffices to show that (286) implies (283). Let  $F(\gamma)$  denote the matrix on the left hand side of (283). Since  $F(\gamma) = F_0 + \gamma F_1 + \gamma^2 F_2$ , where

$$F_2 = \begin{bmatrix} A_1^T Z A_1 & 0 & 0 & A_1^T Z A_{d1} & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & A_{d1}^T Z A_{d1} & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix}$$

and  $F_2 \geq 0$ ,  $F(\gamma)$  is convex matrix function of  $\gamma$ . Using Lemma 7.1, (286) implies (283). ■

### 7.3 Numerical Example

In order to validate the previous theoretical developments, in this section we consider a numerical example motivated by control of chatter during milling. The dynamics depend both on the cutter speed, as well as on the machine tool and piece contact geometry. The force acting on the tool is a function not only of the current displacement of the tool, but also the surface characteristics, hence the displacement at the previous tool pass. This induces a delay into the system. The force depends also on the angular position of the blade, which plays the role of a time-varying parameter. Fig. 12 depicts the geometry of



**Figure 12:** LPV Time-Delay System of Milling Process

the cutting process, where  $K$  is a time varying parameter. As shown in the figure, for this example, the cutter has two blades that are used to remove the material of the workpiece. The blades are assumed to rotate at a constant speed  $\omega$ . The equations of this system can



be derived directly from Fig. 12 as follows

$$m_1 \ddot{x}_1 + k_1(x_1 - x_2) = f \quad (287a)$$

$$m_2 \ddot{x}_2 + c\dot{x}_2 + k_1(x_2 - x_1) + k_2x_2 = 0 \quad (287b)$$

$$f = k \sin(\phi + \beta)h(t) \quad (287c)$$

$$h(t) = h_{\text{ave}} + \sin(\phi)[x_1(t - \tau) - x_1(t)] \quad (287d)$$

where  $k_1$  and  $k_2$  are the stiffnesses of the two springs,  $c$  is the damping coefficient,  $m_1$  is the mass of the cutter,  $m_2$  is the mass of the spindle. The displacements of the blade and tool are  $x_1$  and  $x_2$  respectively. The angle  $\beta$  depends on the particular material and tool used, and is constant. The angle  $\phi$  denotes the angular position of the blade and  $k$  is the cutting stiffness.  $h_{\text{ave}}$  is the average chip thickness (here assumed, without loss of generality, that  $h_{\text{ave}} = 0$ ) and  $\tau = \pi/\omega$  is the delay between successive passes of the blades. The previous equations can be written as

$$\ddot{x}_1 = \frac{1}{m_1}[-k_1x_1 + k_1x_2 - k \sin(\phi + \beta) \sin(\phi)x_1 + k \sin(\phi + \beta) \sin(\phi)x_1(t - \tau)] \quad (288a)$$

$$\ddot{x}_2 = \frac{1}{m_2}[k_1x_1 - k_1x_2 - k_2x_2 - c\dot{x}_2] \quad (288b)$$

or in state-space form,

$$\dot{X}(t) = A(\phi)X(t) + A_d(\phi)X(t - \tau) \quad (289)$$

where  $X = [x_1, x_2, \dot{x}_1, \dot{x}_2]^T$  and

$$A(\phi) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1 + k \sin(\phi) \sin(\phi + \beta)}{m_1} & \frac{k_1}{m_1} & 0 & 0 \\ \frac{k_1}{m_2} & -\frac{k_1 + k_2}{m_2} & 0 & -\frac{c}{m_2} \end{bmatrix}, \quad A_d(\phi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k \sin(\phi) \sin(\phi + \beta)}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

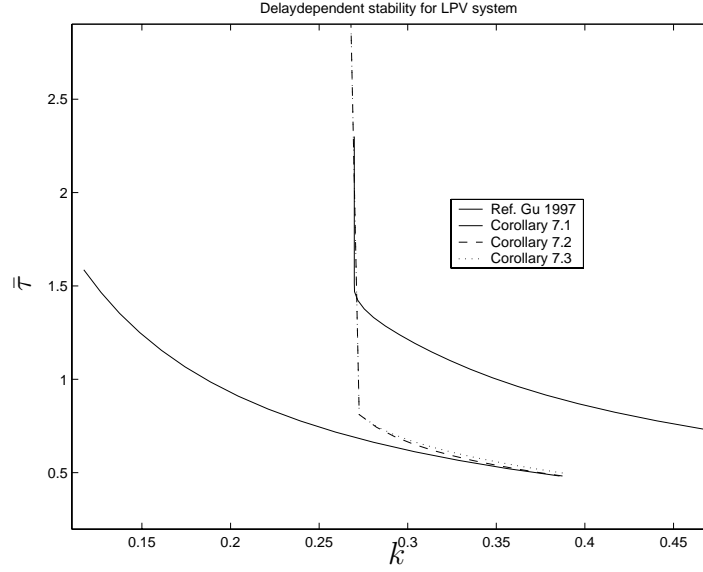
Assume that the problem data are same with that in reference [90], i.e.  $m_1 = 1$ ,  $m_2 = 2$ ,  $k_1 = 10$ ,  $k_2 = 20$ ,  $c = 0.5$ ,  $\beta = 70$  deg. Since

$$\sin(\phi) \sin(\phi + \beta) = \frac{1}{2}[\cos(\beta) - \cos(2\phi + \beta)] = 0.1710 - 0.5 \cos(2\phi + \beta)$$

the system matrices take the form  $A(\gamma) = A_0 + \gamma A_1$  and  $A_d(\gamma) = A_{d0} + \gamma A_{d1}$ , where  $\gamma = \cos(2\phi + \beta) \in [-1, 1]$  and

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(10 + 0.1710k) & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, & A_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ A_{d0} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1710k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & A_{d1} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (290)$$

We have formulated the process as an LPV time-delayed system which is also parametrically dependent on the cutting stiffness  $k$  (constant in this example). Our objective is to find, for a given angular velocity  $\omega$  of the blades, the value of the cutting stiffness  $k$ , such that this LPV system will be stable for all  $\gamma(t) \in [-1, 1]$  and delay  $\tau = \pi/\omega$ . In addition, we wish to find (for a given cutting stiffness  $k$  and speed  $\omega$ ) the maximal allowable time-delay  $\bar{\tau}$ . If  $\pi/\omega < \bar{\tau}$ , the system will be stable. Several tests were performed, using the results of Corollaries 7.1-7.3. The results of our analysis are shown in Fig. 13. These results indicate that the stability conditions of Corollary 7.2 and Corollary 7.3 are better than the one of Corollary 7.1. This is because the system in (254), which is used in Corollary 7.1, is a special case of the system in (255), which is used in Corollaries 7.2 and 7.3. Moreover, the Lyapunov functionals used in Corollaries 7.1 and 7.2 are both special cases of the functional in Corollary 7.3. It is therefore expected that Corollary 7.3 should give a less conservative stability condition than Corollaries 7.1 and 7.2. This is verified by the calculation results shown in Fig. 13. From the same figure, Corollaries 7.2 and 7.3 seem to provide an almost “delay-independent” stability condition for  $k < 0.275$ . For comparison, we also applied the results of [30] to this example. The results of [30] assume however that the delay interval is known and thus, are not readily applicable to the calculation of the maximum delay interval. A bisection method was used to calculate  $\bar{\tau}$  for each value of  $k$  in this case. On the contrary, the maximum delay from Corollaries 7.1-7.3 can be calculated directly, since



**Figure 13:** Maximal Allowable Time Delay Predicted by Corollaries 7.1-7.3

these conditions can be cast as generalized eigenvalue problems [17]. Using three discrete elements, the result from [30] is shown as the solid line in Fig. 13. The prediction for the stability region in this case is less conservative than the one predicted by Corollaries 7.1-7.3. However, the computer CPU time was an order of magnitude larger than the one required for the stability tests of Corollaries 7.1-7.3.

We also applied the delay-independent stability conditions of Theorems 7.2-7.6. Note that for the milling example investigated here,  $A_2 = A_{d2} = 0$  and  $\dot{\gamma} \in (-\infty, \infty)$ . The system is delay-independent stable if  $k \leq K_m$ . Table 7.3 summarizes the results of the tests that ensure delay-independent stability. The more general the Lyapunov matrices  $Q(\gamma)$  and  $P(\gamma)$ , i.e. the more freedom we have to choose the matrices  $Q(\gamma)$  and  $P(\gamma)$ , the less conservative results can be expected. This can be illustrated by choosing different forms for the matrix  $Q(\gamma)$  while applying Theorem 7.2. As stated in Remark 7.2 and Remark 7.3, for the LPV time-delay systems in which the system matrices  $A(\gamma)$  and  $A_d(\gamma)$  depend affinely on  $\gamma$  and the parameter variation rate  $\dot{\gamma}$  is unbounded, Theorem 7.5 and Theorem 7.6 give the same results, which is also the same with that of Theorem 7.2 in case of choosing  $Q(\gamma)$  to dependent affinely on  $\gamma$ .

**Table 1:** Results for delay-independent stability. The system is delay-independent stable if  $k \leq K_m$ .

Theorem	$K_m$	Notes
Th. 7.2	0.2671	$Q(\gamma) = Q_0$
Th. 7.2	0.2695	$Q(\gamma) = Q_0 + \gamma Q_1$
Th. 7.2	0.3043	$Q(\gamma)$ , gridding
Th. 7.3	N/A	$A_2 = 0$
Th. 7.4	same as Th. 7.2	$P = P_0$ , constant
Th. 7.5	0.2695	Remark 7.2
Th. 7.6	0.2695	Remark 7.3

Finally, we emphasize that all results shown in Fig. 13 are conservative. The maximal value of the delay that ensures stability is not known for this example.

## 7.4 Conclusions

In this chapter, compared to the existing results in the literature on the stability conditions for LTI time-delayed systems and LPV systems, we have developed stability tests for Linear Parameter Varying (LPV) systems subject to delays. Both delay-independent and delay-dependent stability conditions are derived using properly chosen Lyapunov-Krasovskii functionals. These conditions are expressed in terms of LMIs and are thus computationally tractable. Delay-independent criteria ensure that stability is maintained for all parameter values and every value of the time-delay. The delay-dependent tests incorporate explicitly the maximum allowable delay before loss of stability. Using gridding of the parameter space both delay-dependent and delay-independent stability tests can be cast as convex optimization problems involving LMI's, which can be solved efficiently using current computer software [28]. A numerical example motivated by the problem of machine milling is used to compare the developed stability analysis tests.

In the delay-independent case, bounds on the parameter variation can also be incorporated using Lyapunov-Krasovskii functionals. The ensuing stability tests assume that  $\gamma(t)$ ,  $\gamma(t-\tau)$  and  $\dot{\gamma}(t)$  are independent. The independence of  $\gamma(t)$  and  $\gamma(t-\tau)$  is a reasonable assumption for delay-independent stability ( $\bar{\tau} \rightarrow \infty$ ) and for no variation bounds on  $\gamma$ . As the bound

on  $\dot{\gamma}$  becomes increasingly smaller the conservativeness of the results increases, as  $\gamma(t)$  and  $\gamma(t - \tau)$  cannot be treated as independent. Delay-dependent stability tests may be more appropriate in this case. Our delay-dependent results do not incorporate any parameter variation bounds. Nonetheless, it is expected that for several problems, delay-dependent results may not be very conservative even if they do not incorporate explicitly any bounds on the parameter variation. This is because any such bounds may be implicit in the maximum allowable delay.

Apart from the recent article of [82], the developments of the present work are the only known results for LPV time-delayed systems. The results presented herein follow closely the corresponding results developed for LTI time-delayed systems [47, 61, 89]. Additional results for LTI time-delayed systems have been developed by Gu [30, 31] using a discretization scheme. Although these results are necessary and sufficient for the LTI case, they are only sufficient for the uncertain LTI and LPV cases. In addition, they are computationally expensive, and they are not directly extendable to synthesis. Although in this work we only address the stability of LPV time-delayed systems free of disturbances, it should be mentioned that the results can be easily extended to the analysis of time-delayed LPV systems satisfying an  $H_\infty$  bound following an approach similar to the one in [60, 47, 70, 82].

## CHAPTER VIII

### CONCLUSIONS AND FUTURE WORK

#### 8.1 *Conclusions*

In this dissertation, we developed several stability conditions for linear dynamic systems, including linear parameter-varying (LPV), time-delay systems, slow LPV systems, and linear time invariant parameter-dependent(LTIPD) systems. These stability conditions are less conservative and/or in computable LMIs. The contributions of this dissertation are listed as follows.

First, the complete, exact stability domain for single-parameter LTIPD systems is synthesized by extending existing results, which can only give one stability interval over  $\mathbb{R}$  even though the whole stability domain could be one interval or a union of several disjointed interval over  $\mathbb{R}$ . This domain is calculated through a guardian map which involves the determinant of the Kronecker sum of a matrix with itself. This approach is then improved through a new defined guardian map involving the bialternate sum of a matrix with itself, which needs less computation compared to the Kronecker sum. The method to calculate the whole stability domain for single-parameter LTIPD systems is then generalized for multi-parameter LTIPD systems.

Second, a class of parameter-dependent Lyapunov functions is proposed, which can be used to assess the stability properties of single-parameter LTIPD systems in a non-conservative manner. It is shown that stability of LTIPD systems is equivalent to the existence of a Lyapunov function of a polynomial type (in terms of the parameter) of known, bounded degree. For the system matrix of dimension  $n \times n$ , this bound of polynomial degree of the Lyapunov functions is then reduced from  $n^2 - 1$  to  $\frac{1}{2}n(n+1) - 1$  by taking the advantage that

the Lyapunov matrices are symmetric. If the matrix multiplying the parameter is not full rank, the polynomial order can be reduced even further. It is also shown that checking the feasibility of the two Lyapunov matrix inequalities over a compact set can be cast as a convex optimization problem. Therefore the nonconservative stability conditions can be cast as two LMIs. Such Lyapunov functions and LMI, nonconservative stability conditions for affine single-parameter LTIPD systems are then generalized to single-parameter polynomially-dependent LTIPD systems, and affine multi-parameter LTIPD systems.

Third, we provide a computationally tractable criteria for analyzing the stability of LPV *time-delayed* systems. Both delay-independent and delay-dependent stability conditions are achieved, which are derived using appropriately selected Lyapunov-Krasovskii functionals. According to the system parameter dependence, these functionals can be selected to obtain increasingly non-conservative results. Gridding techniques may be used to cast these tests as Linear Matrix Inequalities (LMI's). In cases when the system matrices depend affinely or quadratically on the parameter, gridding could be avoided.

## **8.2 Future Work**

Due to the engineering need and the special structure of the particular slow LPV systems treated in this work, the stability analysis and controller synthesis of these systems is still under active research. We believe that our stability analysis results, especially the stability conditions expressed in terms of LMIs, are extendable to synthesis. Some suggestions for future research are outlined below.

For the controller synthesis problem of parameter-dependent LTI systems or linear parameter varying (LPV) systems, gain-scheduling techniques have been the subject of research over recent years [2, 3, 5, 4, 28, 10, 9]. A key point in the characterization of gain-scheduled controllers is the search for adequate Lyapunov functions that establish stability and a performance bound for the closed-loop system. The so-called quadratic gain-scheduled

techniques make use of a fixed Lyapunov function, which is not dependent on the scheduled variables, in order to characterize stability and performance. These approaches are potentially very conservative. Chapter 4 suggests a polynomial type parameter-dependent Lyapunov function for robust stability analysis of parameter-dependent LTI systems. The stability condition, which is derived using such Lyapunov functions and is expressed in terms of LMIs, is necessary and sufficient. It is desired to also use such kind of Lyapunov function to develop an algorithm for gain-scheduled controller synthesis and achieve nonconservative results for parameter-dependent LTI systems.

The following plant  $G(\rho)$  is a parameter-dependent LTI system or a slow LPV system.

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2 \\ C_1(\rho) & D_{11}(\rho) & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \quad (291)$$

where,  $x(t) \in \mathbb{R}^n$  is the state variable vector,  $w(t) \in \mathbb{R}^{n_w}$  is the exogenous disturbance input vector,  $u(t) \in \mathbb{R}^{n_u}$  is the control input vector, the plant outputs are partitioned into the controlled output  $z(t) \in \mathbb{R}^{n_z}$  and the measurement output  $y(t) \in \mathbb{R}^{n_y}$ . The system matrices  $A(\rho)$ ,  $B_1(\rho)$ ,  $B_2$ ,  $C_1(\rho)$ ,  $D_{11}(\rho)$ ,  $D_{12}$ ,  $C_2$ ,  $D_{21}$ ,  $D_{22}$  are of proper dimension in the field of  $\mathbb{R}$ . Without loss of generality,  $D_{22} = 0$  is assumed. It is also assumed that the matrices  $B_2$ ,  $D_{12}$ ,  $C_2$  and  $D_{21}$  are constant. The parameter  $\rho \in \mathbb{R}^{n_\rho}$  is not a known priori but it is assumed measurable. A parameter-dependent LTI or a slow LPV system is a special case of the LPV plant with  $\dot{\rho} = 0$  or  $\dot{\rho} \simeq 0$ . In many engineering applications, the system matrices are affine in the parameter  $\rho \in \mathbb{R}$ .

$$\begin{bmatrix} A(\rho) & B_1(\rho) & B_2 \\ C_1(\rho) & D_{11}(\rho) & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} A_0 & B_{10} & B_2 \\ C_{10} & D_{110} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} + \rho \begin{bmatrix} A_1 & B_{11} & 0 \\ C_{11} & D_{111} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (292)$$

In many cases,  $A_1$  is rank deficient, i.e.  $\text{rank}(A_1) < n$ , and the matrices  $B_{11}$ ,  $C_{11}$  and  $D_{111}$  are also rank deficient or even rank 0.

For the plant  $G(\rho)$  in (291) with (292) and  $\dot{\rho} = 0$  or  $\dot{\rho} \simeq 0$ , the following output feedback controller  $K(\rho)$  is desired to ensure internal stability and a minimal  $H_\infty$  or  $L_2$ -gain bound  $\gamma$



for the closed-loop system of  $G(\rho)$  and  $\mathcal{K}(\rho)$  from the disturbance input  $w$  to the performance output  $z$ .

$$\begin{bmatrix} \dot{x}_k \\ u \end{bmatrix} = \begin{bmatrix} A_k(\rho) & B_k(\rho) \\ C_k(\rho) & D_k(\rho) \end{bmatrix} \begin{bmatrix} x_k \\ y \end{bmatrix} \quad (293)$$

For the linear parameter-dependent plant  $G(\rho)$  in (291) with (292) and  $\dot{\rho} = 0$  or  $\dot{\rho} \simeq 0$ , the regular gain-scheduling techniques [28] can only achieve conservative performance since it allows the varying rate of parameter, i.e.  $\dot{\rho}$  to be infinite.

### 8.2.1 Open Problem 1

The future synthesis work for LTI parameter-dependent systems or slow LPV systems, similarly to gain-scheduling techniques, can be based on projection lemma [26, 17] and the following Bounded Real Lemma (BRL) [80, 99].

**Lemma 8.1** *For the LTI closed-loop system,*

$$\left\| \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \right\|_{\infty} < \gamma \quad (294)$$

*if and only if there exists a storage function  $P_{cl} = P_{cl}^T$  such that*

$$\begin{bmatrix} A_{cl}^T P_{cl} + P_{cl} A_{cl} & * & * \\ B_{cl}^T P_{cl} & -\gamma I & * \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 \quad (295)$$

When  $A_{cl}$ ,  $B_{cl}$ ,  $C_{cl}$ ,  $D_{cl}$  are parameter-dependent,  $P_{cl}$  in (295) should be also parameter-dependent such that condition (295) is still necessary and sufficient for condition (294). If  $P_{cl}$  in (295) is assumed to be constant, condition (295) is only sufficient for condition (294) and performance  $\gamma$  is potentially conservative. It is of interest to search for a parameter-dependent Lyapunov function  $P_{cl}(\rho)$  such that condition (295) is necessary and sufficient for condition (294) even for parameter-dependent  $A_{cl}(\rho)$ ,  $B_{cl}(\rho)$ ,  $C_{cl}(\rho)$ ,  $D_{cl}(\rho)$ .

In most engineering cases, the plant model is linearly dependent on parameter  $\rho$  as in (292).

If the controller is also linearly dependent on parameter  $\rho$ , i.e.,

$$\begin{bmatrix} A_k(\rho) & B_k(\rho) \\ C_k(\rho) & D_k(\rho) \end{bmatrix} = \begin{bmatrix} A_{k0} & B_{k0} \\ C_{k0} & D_{k0} \end{bmatrix} + \rho \begin{bmatrix} A_{k1} & B_{k1} \\ C_{k1} & D_{k1} \end{bmatrix} \quad (296)$$

the closed-loop system will be linearly dependent on parameter  $\rho$  since

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} & B_{cl} &= \begin{bmatrix} B_1 + B_2 D_k D_{21} \\ B_k D_{21} \end{bmatrix} \\ C_{cl} &= \begin{bmatrix} C_1 + D_{12} D_k C_2 & D_{12} C_k \end{bmatrix} & D_{cl} &= \begin{bmatrix} D_{11} + D_{12} D_k D_{21} \end{bmatrix} \end{aligned}$$

Thus, we only need to search Lyapunov function  $P(\rho)$  for the case that  $A_{cl}(\rho)$ ,  $B_{cl}(\rho)$ ,  $C_{cl}(\rho)$ ,  $D_{cl}(\rho)$  are linearly related to  $\rho$ .

### 8.2.2 Open Problem 2

If  $A_1$  of the plant model in (292) is rank deficient, the controller's parameter-dependent part  $A_{k1}$  can not be guaranteed to be rank deficient with the regular gain-scheduling techniques. If  $A_{k1}$  is of lower rank, the controller will be easier to realize. We believe it is possible to design a linear parameter dependent controller as in (296) with its matrix  $A_{k1}$  of lower rank that can still achieve a satisfactory performance for the closed-loop system.

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